Consistent truncation of $d = 11$ supergravity on $AdS_4 \times S^7$

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Abstract

We study the system of equations derived twenty five years ago by B. de Wit and the first author [Nucl. Phys. B281 (1987) 211] as conditions for the consistent truncation of eleven-dimensional supergravity on $AdS_4 \times S^7$ to gauged $\mathcal{N} = 8$ supergravity in four dimensions. By exploiting the $E_{7(7)}$ symmetry, we determine the most general solution to this system at each point on the coset space $E_{7(7)}/SU(8)$. We show that invariants of the general solution are given by the fluxes in eleven-dimensional supergravity. This allows us to both clarify the explicit non-linear ansätze for the fluxes given previously and to fill a gap in the original proof of the consistent truncation. These results are illustrated with several examples.

Contents

1 Introduction

Among the known examples of consistent non-linear embeddings in Kaluza-Klein supergravity (and Kaluza-Klein theories in general), the non-linear embedding of $\mathcal{N}=8$, $d=4$ gauged supergravity [1] into $\mathcal{N}=1, d=11$ supergravity [2] stands out as the most subtle and complicated. This embedding was derived a long time ago in [3] on the basis of the SU(8) invariant reformulation of $d=11$ supergravity presented in [4] (a list of references to earlier work can be found in [3]). However, with the exception of [5], where the simpler embedding of maximal $d=7$ gauged supergravity into the $d=11$ theory was completely worked out, there has not been much follow-up work on maximal supergravity embeddings since then. In particular, no complete proof exists for the $AdS_5 \times S^5$ compactification of IIB supergravity to maximal gauged $\mathcal{N}=8$ supergravity in $d=5$, although partial formulae for the embedding were obtained in [6–8]. By contrast, there has

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been considerable work on consistent truncations of $\mathcal{N}=1$, $d=11$ supergravity to non-maximal supergravities in $d=4$, whose scalar sectors are much simpler.[1]

In this paper we re-analyze the embedding of the scalar sector of the $\mathcal{N}=8$ theory and, in particular, examine the flux ansätze in [3] for solutions of $\mathcal{N}=1$, $d=11$ supergravity that correspond to lifts of critical points of the scalar potential in four dimensions. Our present interest in this problem has been motivated on the one hand by the recent discovery of a large number of new critical points [18–21] for which the corresponding eleven-dimensional solutions are not yet known. On the other hand, the explicit flux formulae in [3] have never been tested for any but the maximally supersymmetric point. The present investigation originated from an attempt to extend this analysis to non-trivial vacua, and to test the general formulae by performing numerical checks for some configurations of the scalar fields. To our surprise, these checks revealed systematic inconsistencies.[2] This raised questions not just about the flux formulae *per se*, but also about the completeness of the proof of consistency for the S^7 truncation in [3]. In this paper, we resolve the apparent discrepancies and complete the proof of consistency, on the way also deriving the exact formulae for the non-linear flux ansätze. Indeed, the formulae given in [3], after considerable work, turn out to be essentially correct, *modulo* an important subtlety that was not appreciated there, and which is the main subject of the present paper.

Recall that solutions of $d=11$ supergravity corresponding to given critical points of gauged $\mathcal{N}=8$ supergravity are warped products $AdS_4 \times \mathcal{M}_7$,

$$ds_{11} = \Delta^{-1}ds^2_{AdS_4} + ds^2_{\mathcal{M}_7}\,, \tag{1.1}$$

$$F_{(4)} = f\,\Delta^{-2}\,\mathrm{vol}_{AdS_4} + \tfrac{1}{4!}\,F_{abcd}\,e^a \wedge e^b \wedge e^c \wedge e^d\,, \tag{1.2}$$

where $ds^2_{AdS_4}$ denotes the line element in AdS_4 with warp factor Δ^{-1}, and $ds^2_{\mathcal{M}_7} \equiv g_{mn}\,dy^m \otimes dy^n$ is the internal seven-metric (as in [3], we will label four-dimensional coordinates x^μ by Greek indices $\mu, \nu... = 0, 1, 2, 3$, and internal coordinates y^m by Latin indices $m, n, ... = 1, ..., 7$). The flux components are defined in the usual way, with *flat* indices and $24if \equiv \varepsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta\gamma\delta}$ (so the constant $f_0 \equiv f\Delta^{-2}$ is the Freund-Rubin parameter [23]). Hence, the internal manifold $\mathcal{M}_7$ is a deformation of the seven-sphere S^7 (which corresponds to the maximally supersymmetric vacuum [24]). In fact, such deformations can be studied for *any* field configuration of the $d=4$ theory satisfying the field equations of $\mathcal{N}=8$ supergravity, in which case the internal metric and fluxes depend on both x and y, such as for instance the AdS_4-type vacua with x-dependent scalar field configurations which have attracted recent interest in the context of M2-branes and holographic superconductors. The main question then concerns (*i*) *how* to construct the non-linear embedding of a given $d=4$ configuration into the $d=11$ theory, and (*ii*) the consistency

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[1]For a partial list of references, see [9–17].
[2]Recently, a similar inconsistency for the SO(7)$^+$ point was independently observed in [22].

of this embedding. By a *consistent truncation* (or embedding) we shall here generally mean that any $d=4$ solution (whether x-independent or not), when embedded into the $d=11$ theory, should yield an exact solution of the latter *at the full non-linear level* (see e.g. [25] for an introductory review). As we will see, this requirement will lead to rather complicated formulae for both the internal metric and the fluxes in terms of the $d=4$ fields.

2   Synopsis

In this section we recall some central results from earlier work and summarize our main new insights. We strongly recommend that readers consult the two main references [3, 4], whose notations and conventions we will follow throughout this paper, as well as [26] for further details whenever necessary.

The explicit construction of the lift in [26, 4, 3] starts with the so-called *generalized vielbein*. This object is a 'soldering form' with one internal upper world index and two flat (tangent space) SU(8) indices, and plays a key role in the SU(8) invariant reformulation of $d=11$ supergravity presented in [27, 4]. It is expressed in two different and independent ways, one coming from the $d=11$ side via the reformulation [4], and the other coming from the $d=4$ theory, and in terms of the scalar fields of $\mathcal{N}=8$ supergravity and the S^7 Killing spinors. The comparison between the $d=4$ and the $d=11$ expressions, obtained by judicious analysis of the supersymmetry variations, then yields crucial information about the non-linear embedding, as we now explain.

Let us start with the $d=11$ side, which is based on the reformulation [4], where the original tangent space symmetry SO(1, 10) is replaced by SO(1, 3) × SU(8), as appropriate to a (4+7)-decomposition of the original theory [2], and where the dependence on all coordinates is initially retained. The generalized vielbein is defined from the $d=11$ supersymmetry variations as

$$e^m_{AB}(x,y) = i\, e_a{}^m\, \Delta^{-1/2}\, \Gamma^a_{AB}\,, \tag{2.1}$$

where $e_m{}^a(x,y)$ is the siebenbein of the full metric on $\mathcal{M}_7$, and $e_a{}^m(x,y)$ its inverse. The factor Δ is essentially the siebenbein determinant, except that for convenience we define

$$S_a{}^b(x,y) \;\equiv\; \mathring{e}_a{}^m(y) e_m{}^b(x,y)\,, \qquad \Delta \;\equiv\; \det S\,, \tag{2.2}$$

thus taking out the y-dependent background factor $\det \mathring{e}_m{}^a$, where $\mathring{e}_m{}^a(y)$ is the background S^7 siebenbein, and the metric on the round S^7 is $\mathring{g}_{mn} = \mathring{e}_m{}^a \mathring{e}_n{}^b \delta_{ab}$.[3] The SO(7) gamma matrices, Γ^a, are purely imaginary, and therefore e^m_{AB}, as defined in (2.1), is real. However, a crucial step

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[3]Factoring out the background S^7 siebenbein leads to extra determinant factors $\mathring{g}$ in various formulae below. Such factors can be dropped for all practical purposes by adopting a local frame where $\mathring{e}_m{}^a = \delta_m{}^a$.

taken in [4] in order to re-write the theory into SU(8) covariant form is now to replace (2.1) by the more general definition

$$e^m_{AB}(x,y) = i\, e_a{}^m\, \Delta^{-1/2}\, (\Phi^T \Gamma^a \Phi)_{AB}\,, \quad e^{mAB} = (e^m_{AB})^*\,, \tag{2.3}$$

where $\Phi^A{}_B(x,y)$ is an arbitrary *local* SU(8) rotation depending on all eleven coordinates. In this way the local SO(7) tangent space symmetry is enhanced to local SU(8) in eleven dimensions. As a consequence, the *real* internal siebenbein is converted into a *complex* object transforming under local SU(8), unlike the original siebenbein which transforms only under SO(7). The real form (2.1) is then viewed as an SU(8) tensor $\mathbf{28} \oplus \overline{\mathbf{28}}$ taken in a special gauge.[4] This gauge choice will prove extremely useful below, and we will return to it on several occasions.

On the $d=4$ side, by contrast, the generalized vielbein is determined in terms of the 70 scalar fields of $\mathcal{N}=8$ supergravity and the 28 Killing vectors on S^7 as

$$e^m_{ij}(x,y) = K^{m\,IJ}(y)\left(u_{ij}{}^{IJ}(x) + v_{ijIJ}(x)\right), \qquad e^{m\,ij} = (e^m_{ij})^*\,, \tag{2.4}$$

where the scalar '56-bein'

$$\mathcal{V}(x) = \begin{pmatrix} u_{ij}{}^{IJ}(x) & v_{ijIJ}(x) \\ v^{ijIJ}(x) & u^{ij}{}_{IJ}(x) \end{pmatrix} \in \mathrm{E}_{7(7)}\,, \tag{2.5}$$

is an element of the maximally ('split') non-compact form of the E_7 Lie group in the fundamental representation. The Killing vectors in (2.4) are represented in the usual way as bilinears of the Killing spinors,

$$K^{m\,IJ} = i\,\mathring{e}_a{}^m\,\bar{\eta}^I \Gamma^a \eta^J\,. \tag{2.6}$$

Those (commuting) Killing spinors, $\eta^I(y)$, satisfy

$$\left(\mathring{D}_m + \tfrac{i}{2}\,m_7\,\mathring{\Gamma}_m\right)\eta^I = 0\,, \qquad I = 1,\ldots,8\,, \tag{2.7}$$

where m_7 is the inverse radius of S^7, $\mathring{\Gamma}_m \equiv \mathring{e}_m{}^a \Gamma_a$, and $\mathring{D}_m$ denotes the S^7 background covariant derivative.

We also note that, when considered as 8×8 matrices, the Killing spinors are orthonormal, in the sense that $\eta^I{}_A \eta^A{}_J = \delta^I{}_J$, etc. As explained in [3], this allows us to use the Killing spinors to convert the two kinds of SU(8) indices: $A, B, C, \ldots$, and $i, j, k, \ldots$ or $I, J, K, \ldots$, appropriate to $d=11$ and $d=4$, respectively, into one another. However, a direct comparison between $d=11$ and $d=4$ quantities is more subtle, and realizing that was one of the crucial steps in the proof of the consistent truncation in [3].

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[4]In fact, the complex pair (e^m_{AB}, e^{mAB}) can be assigned to the $\mathbf{56}$ representation of $\mathrm{E}_{7(7)}$, even though the latter is only a symmetry of the theory when compactified on $\mathbb{R}^{1,3} \times T^7$ [28].

One key ingredient in the proof of consistency is the SU(8) rotation matrix $\Phi \equiv \Phi(x,y)$ introduced in (2.3), which is required for a consistent 'alignment' of the $d{=}4$ and $d{=}11$ theories and, in particular, of the generalized vielbeine (2.3) and (2.4). More precisely, the $d=4$ and $d{=}11$ vielbeine (2.3) and (2.4) above are related by

$$e^m_{AB}(x,y) \equiv i\, e_a{}^m\, \Delta^{-1/2}\, \Gamma^a_{CD}\, \Phi^C{}_A \Phi^D{}_B = e^m_{ij}\, \eta^i{}_A \eta^j{}_B\,. \tag{2.8}$$

This formula makes obvious the necessity of complexifying the original internal siebenbein (2.1) via (2.3), because (2.4) and hence the right hand side of (2.8) are manifestly complex. It also confirms that the SU(8) rotation Φ in general depends non-trivially on *both* the $d{=}4$ (space-time) and the $d{=}7$ (internal) coordinates. The existence of the SU(8) rotation Φ for *any* vielbein of the form (2.4) follows from the fact that the latter can be shown to satisfy the Clifford property by virtue of some $E_{7(7)}$ identities, as explained in Section 2 of [3].

From the two different representations of the generalized vielbein in (2.1) and (2.4), and from (2.8), we deduce two key results:

(i) The *non-linear metric ansatz* [26]

$$8\,(\Delta^{-1}\, g^{mn})(x,y) = e^m_{ij} e^{n\,ij} = (K^{mIJ} K^{nKL})(y)(u_{ij}{}^{IJ} + v_{ijIJ})(u^{ij}{}_{KL} + v^{ijKL})(x)\,, \tag{2.9}$$

implicitly giving the dependence of the internal metric on the scalar 56-bein $\mathcal{V}(x)$ and the S^7 Killing vectors $K^{IJ}(y)$. Given any configuration of the $\mathcal{N}{=}8$ fields, this formula can be solved (at least in principle) for the embedded internal metric $g_{mn}(x,y)$.

(ii) The SU(8) rotation matrix $\Phi(x,y)$ is determined as a function of the scalar 56-bein and the S^7 Killing spinors. As already mentioned, this SU(8) rotation is needed to 'align' the linear fermionic ansätze with the non-linear bosonic ones in an SU(8) gauge where the linear fermionic ansätze are exact to all orders, as explained in [3].

Of course, closed form expressions for either $g_{mn}(x,y)$ and $\Phi(x,y)$ are hard to come by, and can only be obtained in very special circumstances. Nevertheless, the non-linear metric ansatz (2.9) has been successfully tested over the years for a variety of non-trivial solutions: critical points [26, 29, 30], RG flows [29, 31–34] and quadratic fluctuations [35]. It was also used to construct smaller truncations [11, 35], and was generalized to maximal supergravities in $d{=}5$ [6] and $d{=}7$ [5, 36]. Observe that (2.9) fixes the overall normalization of the metric relative to the trivial vacuum for *any* solution of the $d{=}11$ equations of motion corresponding to a consistent embedding of an on-shell configuration of $N{=}8$ supergravity (whereas the former are in principle only determined up to an overall scaling).

Similarly, the SU(8) rotation is known in closed form only for some special critical points, the very simplest example being the maximally supersymmetric point, $\Phi = 1$. Corrections to

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first order in the supersymmetry parameter induced by the non-linear embedding were given in [37], although without mention of SU(8). For purely scalar fluctuations, with no pseudoscalars, a perturbative expansion for Φ was derived in [38], but no closed form for the summed series is known. For all other scalar and pseudoscalar configurations, the explicit solutions for Φ become rapidly very complicated and cumbersome, as can be seen from the examples in Section 6.

While the above results are enough to derive the non-linear metric ansatz, they are not sufficient to obtain the fluxes as functions of the scalar 56-bein. For this we need to invoke the extra information provided by two consistency requirements, namely (i) the generalized vielbein postulate, and (ii) the so-called $\mathfrak{A}$-equations.[5]

The first condition derived in [4] is that the generalized vielbein must satisfy an equation, the so-called *Generalized Vielbein Postulate* (or *GVP*, for short). Like the generalized vielbein itself, this condition comes in two different guises. Again, we first discuss the equation as obtained from the $d=11$ side where e^m_{AB} must obey

$$\mathring{D}_m e^n_{AB} + \mathcal{B}_m{}^C{}_{[A} e^n_{B]C} + \mathcal{A}_{m\,ABCD} e^{n\,CD} = 0 \,, \tag{2.10}$$

with a corresponding equation for the complex conjugate vielbein e^{mAB}. Here, $\mathcal{B}_m{}^A{}_B(x,y)$ and $\mathcal{A}_{m\,ABCD}(x,y)$ together can be viewed as an $\mathrm{E}_{7(7)}$ connection in the seven internal dimension. The *GVP* constitutes the analog of the corresponding conditions for the $d=4$ connection $(\mathcal{B}_\mu{}^A{}_B\,,\,\mathcal{A}_{\mu ABCD})$, which upon compactification on S^7 reduce to the Cartan-Maurer equations for $\mathrm{E}_{7(7)}$ (with SO(8) gauge covariant derivatives, cf. [3]).

Explicit expressions in terms of $d=11$ fields are obtained by careful analysis of the $d=11$ supersymmetry variations [4]:

$$\mathcal{B}_m{}^A{}_B = \tfrac{1}{2}\,(S^{-1}\mathring{D}_m S)_{ab}\Gamma^{ab}_{AB} + \tfrac{i\sqrt{2}}{14}\,f\,e_{ma}\Gamma^a_{AB} - \tfrac{\sqrt{2}}{48}\,e_m{}^a F_{abcd}\Gamma^{bcd}_{AB}\,, \tag{2.11}$$

$$\mathcal{A}_{mABCD} = -\tfrac{3}{4}\,(S^{-1}\mathring{D}_m S)_{ab}\Gamma^a_{[AB}\Gamma^b_{CD]} + \tfrac{i\sqrt{2}}{56}\,e_{ma}f\,\Gamma^{ab}_{[AB}\Gamma^b_{CD]} + \tfrac{\sqrt{2}}{32}\,e_m{}^a F_{abcd}\Gamma^b_{[AB}\Gamma^{cd}_{CD]}\,. \tag{2.12}$$

The matrix $S_a{}^b$ was already defined in (2.2), while the remaining flux terms originate from the four-form field strength of $d=11$ supergravity in the standard way. These two formulae are only valid in the 'real vielbein gauge' (2.1), and they will change when we switch to another SU(8) gauge. On the other hand, when reverting from a general SU(8) covariant expression back to the real gauge, one must, of course, ensure that the resulting expressions preserve the tensor

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[5]Let us, however, emphasize that the final formulae for the non-linear flux ansätze can only be valid *on-shell* because the dualizations needed to convert the two-form fields from $d=11$ supergravity to scalar fields necessarily require the equations of motion. This is in marked contrast to the $AdS_7 \times S^4$ truncation of Ref. [5], where the scalar fields arise directly in the reduction without dualizations, whereas similar complications can be anticipated for the $AdS_5 \times S^5$ truncation which requires the dualization of a three-form field.

structure inherited from $d=11$ supergravity, which is manifest in (2.11) and (2.12).[6]

An important feature of the real gauge (2.1), not spelled out in [4], is that (2.11) is in fact *not* the most general solution; rather, the *GVP* will still be satisfied if we replace

$$f e_{ma} \;\to\; e_m{}^a X_{a|b}\,, \qquad e_m{}^a F_{abcd} \;\to\; e_m{}^a X_{a|bcd}\,, \tag{2.13}$$

where $X_{a|b}$ is an arbitrary matrix, and $X_{a|bcd}$ is anti-symmetric only in the indices $[bcd]$. In other words, the *GVP* admits solutions which in general are not compatible with the properties of the fluxes dictated by $d=11$ supergravity. A crucial requirement for the consistency of the truncation is therefore to ensure that the embedding formula for the fluxes respects these properties. We will call it the *correct tensor structure* condition.

Let us now turn to the $d=4$ side of the story. Here we have formally the same *GVP* equation

$$\mathring{D}_m e^n_{ij} + \mathcal{B}_m{}^k{}_{[i}\, e^n_{j]k} + \mathcal{A}_{m\,ijkl}\, e^{n\,kl} = 0\,, \tag{2.14}$$

but where the SU(8) gauge field $\mathcal{B}_m{}^i{}_j$ and the self-dual tensor field $\mathcal{A}_{m\,ijkl}$ are now to be expressed in terms of the $d=4$ fields. To match the $d=11$ supersymmetry variations with those of gauged $\mathcal{N}=8$ supergravity, however, we must invoke a second consistency requirement. This is to require that the $\mathfrak{A}_1$ and $\mathfrak{A}_2$ tensors, defined as [3]

$$\mathfrak{A}_1^{ij} = -\tfrac{\sqrt{2}}{4}\left(e^{m\,ik}\,\mathcal{B}_m{}^j{}_k + \mathcal{A}_m{}^{ijkl} e^m_{kl}\right), \tag{2.15}$$

$$\mathfrak{A}_{2l}{}^{ijk} = -\tfrac{\sqrt{2}}{4}\left(3\,e^{m\,[ij}\mathcal{B}_m{}^{k]}{}_l - 3\,e^m_{pq}\,\mathcal{A}_m{}^{pq[ij}\delta^{k]}{}_l - 4\,\mathcal{A}_m{}^{ijkp} e^m_{pl}\right), \tag{2.16}$$

are equal to the corresponding A_1 and A_2 tensors[7] of $\mathcal{N}=8$, $d=4$ supergravity, which parametrize the g-dependent deformations (Yukawa couplings and scalar potential) from the ungauged theory:

$$\mathfrak{A}_1^{ij} = g\,A_1^{ij}\,, \tag{2.17}$$

$$\mathfrak{A}_{2l}{}^{ijk} = g\,A_{2l}{}^{ijk}\,. \tag{2.18}$$

In the remainder we will refer to these equations as the '$\mathfrak{A}$-equations.'

The role of these two conditions in the consistent truncation is to ensure that the dependence on the internal space drops out in the reduction of the supersymmetry variations of $d=11$ supergravity to four dimensions, and that in the process one recovers the complete supersymmetry transformations of gauged $\mathcal{N}=8$ supergravity. In particular, one should note that while the A_1 and A_2 tensors on the right hand side in (2.17) and (2.18) are functions only of the scalar 56-bein of the $d=4$ theory, hence depend only on x, the $\mathfrak{A}_1$ and $\mathfrak{A}_2$ tensors are hybrid objects that *a priori* depend both on the scalar 56-bein and the internal coordinates.

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[6]In addition, the flux components must satisfy the Bianchi identities. As shown in [4], this is guaranteed by the SU(8) covariant field equations.

[7]Explicit formulae for A_1 and A_2 are given in (4.1) and (4.2) below.

A special solution of the GVP (2.14) and the $\mathfrak{A}$-equations (2.17) and (2.18) was constructed explicitly in [3] in terms of the scalar 56-bein and the Killing vectors on S^7 as follows. Consider

$$\mathcal{B}_m{}^i{}_j(\alpha,\beta) = -\tfrac{2}{3}\,\alpha\,m_7\,K_m{}^{IJ}(u^{ik}{}_{IK}u_{jk}{}^{JK} - v^{ikIK}v_{jkJK})$$
$$- \tfrac{2}{3}\,\beta\mathring{D}_mK_n{}^{[IJ}K^{n\,KL]}(v^{ikIJ}u_{jk}{}^{KL} - u^{ik}{}_{IJ}v_{jkKL})\,, \qquad (2.19)$$

$$\mathcal{A}_{m\,ijkl}(\alpha,\beta) = \alpha\,m_7\;K_m{}^{IJ}\left(v_{ijJK}u_{kl}{}^{IK} - u_{ij}{}^{JK}v_{klIK}\right)$$
$$- \beta\mathring{D}_mK_n{}^{[IJ}K^{n\,KL]}(u_{ij}{}^{IJ}u_{kl}{}^{KL} - v_{ijIJ}v_{klKL})\,, \qquad (2.20)$$

where α and β are arbitrary real parameters (recall that the indices on the Killing vectors are raised with the S^7 background metric, that is, $K^{m\,IJ} \equiv \mathring{g}^{mn}K_n^{IJ}$). Substitution of these expressions into (2.14) shows that the GVP is satisfied provided $\alpha + 4\beta = 1$, leaving a one-parameter family of solutions.

The remaining freedom is then fixed by imposing the $\mathfrak{A}$-equations. This is by no means obvious, but happily, the detailed analysis of [3] shows that the $\mathfrak{A}$-equations do have a solution of the form above and indeed fix the free coefficients uniquely,

$$\alpha = \tfrac{4}{7}\,, \qquad \beta = \tfrac{3}{28}\,, \qquad (2.21)$$

when the gauge coupling constant of the $d=4$ theory is set to the inverse radius of S^7,

$$g = \sqrt{2}\,m_7\,. \qquad (2.22)$$

With these values, one re-obtains the correct four-dimensional expressions, such that all dependence on the internal coordinates drops out on the left hand side of (2.17) and (2.18), as required for consistency. For this reason, we will refer to the solution (2.19) and (2.20) *with the special values* (2.21) as the *standard inhomogeneous solution* of the GVP, and simply denote it by $(\mathring{\mathcal{B}}_m{}^i{}_j, \mathring{\mathcal{A}}_{m\,ijkl})$.

Nevertheless, direct translation of (2.19) and (2.20) into the $d=11$ expressions (keeping track of the SU(8) alignment rotation, see below) leads to apparent discrepancies with $d=11$ supergravity, in the sense that the resulting expressions in general will *not* respect the tensor structure required by (2.11) and (2.12). The main new result of the present paper is to show how this defect can be remedied: namely, the GVP and the $\mathfrak{A}$-equations still leave the freedom to modify the standard inhomogeneous solution by a homogeneous term 'in the kernel of the GVP and the $\mathfrak{A}$-equations,' and *this extra homogeneous contribution is precisely what is needed for the fluxes in* (2.11) *and* (2.12) *to acquire the requisite tensor structure compatible with $d=11$ supergravity.* We show that such a correction exists and is unique for *any* point on the scalar manifold $E_{7(7)}/SU(8)$.

Put another way, given any solution to the GVP (2.14) and the $\mathfrak{A}$-equations (2.17) and (2.18), viewed as a system of linear equations for $\mathcal{B}_m{}^i{}_j$ and $\mathcal{A}_{m\,ijkl}$ on the $d=4$ side, one can, at least

in principle, determine the corresponding solution of the *GVP* (2.10) on the $d = 11$ side by performing the SU(8) gauge transformation:

$$\mathcal{B}_m{}^A{}_B = U^A{}_i \big[\mathcal{B}_m{}^i{}_j + 2\mathring{D}_m \big] U^j{}_B \,,$$

$$\mathcal{A}_{m\,ABCD} = \mathcal{A}_{m\,ijkl} U^i{}_A U^j{}_B U^k{}_C U^l{}_D \,,$$

$$(2.23)$$

where

$$U^A{}_i(x,y) \equiv \Phi^A{}_B(x,y) \eta^B{}_i(y) \,,$$

$$(2.24)$$

involves both the SU(8) alignment matrix $\Phi(x,y)$ and the conversion matrix $\eta(y)$ (*alias* Killing spinor) between the two kinds of SU(8) indices. Note that $\mathcal{B}_m$ transforms as a *bona fide* SU(8) gauge connection with the usual inhomogeneous contribution. There is *a priori* no reason why the standard inhomogenous solution (2.20) and (2.19) would yield the particular solution (2.11) and (2.12) with the correct tensor structure in $d = 11$, and, in fact, as we verify explicitly in Section 6, in general it does not. This means that the proof of the consistent truncation based on that particular solution is incomplete; one must still show that for each point on the $E_{7(7)}/SU(8)$ coset there exists a solution to the linear system (2.14), (2.17) and (2.18) on the $d = 4$ side that does have the correct tensor structure in $d = 11$.

Our strategy will be to look for a correction $(\delta\mathcal{B}_m{}^i{}_j \,, \delta\mathcal{A}_{m\,ijkl})$ to the standard inhomogeneous solution and solving the homogeneous part of the *GVP*, such that

$$\mathcal{B}_m{}^i{}_j = \mathring{\mathcal{B}}_m{}^i{}_j + \delta\mathcal{B}_m{}^i{}_j \quad, \qquad \mathcal{A}_{m\,ijkl} = \mathring{\mathcal{A}}_{m\,ijkl} + \delta\mathcal{A}_{m\,ijkl} \,,$$

$$(2.25)$$

satisfy all consistency conditions. In particular, the 'corrections' $\delta\mathcal{B}_m$ and $\delta\mathcal{A}_m$ must drop out of the $\mathfrak{A}$-equations (hence belong to their 'kernel'), as otherwise the agreement with the $d = 4$ theory would be spoiled! Identifying these 'corrections' is actually a simpler problem than finding a full solution because the tensors $(\delta\mathcal{B}_m{}^i{}_j \,, \delta\mathcal{A}_{m\,ijkl})$ satisfy the homogenous system of equations corresponding to (2.14), (2.17) and (2.18) and transform covariantly under the U-rotation. We will not present a closed form solution (which is available in principle, but very cumbersome), but we do prove that it always exists and is unique. Quite remarkably, it will turn out that the closed form solution is not even needed to extract the non-linear ansätze for the fluxes!

Given a solution to all consistency conditions, the fluxes can be either read off from the expansions (2.11) and/or (2.12), or alternatively from the SU(8)-invariant projection that is summarized in the flux formula (7.5) in [3] [8]

$$\tfrac{4}{7} i f\, g_{n[p} \delta_{q]}{}^m + \tfrac{1}{2} F^m{}_{npq} = -i \tfrac{\sqrt{2}}{480} \Delta^4 \, \epsilon_{pqrstuv}\, e^m_{ij} (e^r \bar{e}^s e^t \bar{e}^u e^v)_{kl} \mathcal{A}_n{}^{ijkl} \,.$$

$$(2.26)$$

---

[8] We correct some typos in the original formula. For clarity of notation we put a bar on the complex conjugate vielbein $e^{mij} \equiv \bar{e}^{mij}$ in the formulae below, whenever the SU(8) indices are not written out.

Let us emphasize once again that in general this equation is inconsistent if $\mathcal{A}_m{}^{ijkl}$ is replaced with the standard one $(= \overset{\circ}{\mathcal{A}}_m{}^{ijkl})$ but, as we will show, there always exists a unique $\delta\mathcal{A}_{m\,ijkl}$ such that (2.26) does hold with (2.25). Solving (2.26) for the flux components, we find

$$f = -\tfrac{\sqrt{2}}{48\cdot 5!}\,\Delta^4\,g^{mu}\,\epsilon_{mnpqrst}\,e_{ij}^n\,(e^{[p}\bar{e}^q e^r\bar{e}^s e^{t]})_{kl}\,\mathcal{A}_u{}^{ijkl}\,, \tag{2.27}$$

and

$$F_{mnpq} = -\tfrac{i}{144}\,\Delta^4\,g_{rw}\,e_{ij}^r(e^{[s}\bar{e}^t e^u\bar{e}^v e^{w]})_{kl}\,\epsilon_{stuw[mnp}\,\mathcal{A}_{q]}{}^{ijkl}\,. \tag{2.28}$$

Note that in order to exploit these equations we must first solve for the full metric g_{mn} from (2.9). As we will explain in detail below, f and F_{mnpq} as given in (2.27) and (2.28) are *invariants of the linear system*. In other words, it does not matter which solution to the linear system (2.14), (2.17) and (2.18) one uses to evaluate them by projecting the right hand side of (2.26) onto the components (2.27) and (2.28): all solutions, in particular the standard inhomogeneous solution (2.19)-(2.21) give the same answer! In this sense [3] contains already the complete result for the fluxes. Besides the analytic examples of Section 6 we will present some non-trivial numerical checks of (2.27) in Section 7.

The rest of this paper is more technical, as we present the details of our calculations. In Section 3 we determine the general solutions for *GVP* (2.14) and (2.10). The $\mathfrak{A}$-equations are included in Section 4. We show that the full linear system is invariant under the natural action of $\mathrm{E}_{7(7)}$ and use this symmetry to determine the general solution for all consistency conditions on the $d{=}4$ and the $d{=}11$ sides. This allows us to complete the proof of the consistent truncation. Explicit flux formulae are rederived in Section 5. In Section 6 we illustrate various point of the construction on two examples, the $\mathrm{SO}(7)^-$ and $\mathrm{SO}(7)^+$ families. Some results of numerical explorations are summarized in Section 7 and we conlude in Section 8.

3 The Generalized Vielbein Postulate (GVP)

We now return to the vielbein equation (2.14) in order to explain the construction in more detail and to work out the most general solution of the *GVP*. For a given vielbein, (2.14) can be viewed as an inhomogenous linear equation for the components of the SU(8) gauge field, $\mathcal{B}_m{}^i{}_j$, and the tensor field, $\mathcal{A}_{m\,ijkl}$. Recall that the generalized vielbein, e_{ij}^m, and its complex conjugate, $e^{m\,ij}$, can be assigned to transform in the **56**-dimensional representation of $\mathrm{E}_{7(7)}$. For a given point on the scalar coset represented by the group element $\mathcal{V}(x)$, cf. (2.5), we can rewrite (2.4) as

$$\begin{pmatrix} e_{ij}^m \\ e^{m\,ij} \end{pmatrix} = \begin{pmatrix} u_{ij}{}^{IJ} & v_{ij IJ} \\ v^{ijIJ} & u^{ij}{}_{IJ} \end{pmatrix} \begin{pmatrix} K^m{}_{IJ} \\ K^{m\,IJ} \end{pmatrix}\,, \tag{3.1}$$

where $K^m_{IJ} \equiv K^{m\,IJ}$, as the Killing vectors are real. Similarly, $\mathcal{B}_m{}^i{}_j$ and $\mathcal{A}_{m\,ijkl}$, together with the complex conjugates, can be assigned to the adjoint representation **133** of $E_{7(7)}$,

$$\begin{pmatrix} \delta_{[i}{}^{[k}\mathcal{B}_{m\,j]}{}^{l]} & \mathcal{A}_{m\,ijkl} \\ \mathcal{A}_m{}^{ijkl} & \delta^{[i}{}_{[k}\mathcal{B}_m{}^{j]}{}_{l]} \end{pmatrix} = \mathcal{V} \begin{pmatrix} \delta_{[I}{}^{[K}B_{m\,J]}{}^{L]} & A_{m\,IJKL} \\ A_m{}^{IJKL} & \delta^{[I}{}_{[K}B_m{}^{J]}{}_{L]} \end{pmatrix} \mathcal{V}^{-1}. \tag{3.2}$$

By performing a purely x-dependent $E_{7(7)}$ rotation[9] by $\mathcal{V}^{-1}(x)$, one transforms the GVP (2.14) into an equation that has no explicit dependence on the scalar 56-bein any more [3],

$$\mathring{D}_m\,K^n_{IJ} + B_m{}^K{}_{[I}K^n_{J]K} + A_{m\,IJKL}K^{n\,KL} = 0. \tag{3.3}$$

This equation can be further simplified if we consider the Killing spinors $\eta^I{}_A$ in (2.6) as a local SO(8) $\subset$ SU(8) transformation on S^7. Taking into account the inhomogeneous term in the transformation of $B_m{}^I{}_J$, cf. (2.23), we can also remove the explicit dependence on the S^7 coordinates in (3.3),

$$B_m{}^C{}_{[A}\Gamma^n_{B]C} + A_{m\,ABCD}\Gamma^n_{CD} = 0, \tag{3.4}$$

ending up with a homogenous equation for $B_m{}^A{}_B$ and $A_{m\,ABCD}$. This amounts to analyzing (3.3) at the North Pole of S^7, from where it can be parallel transported back to any other point by application of the matrix η. The anti-hermitean $B_m{}^A{}_B$ and the complex-selfdual $A_{m\,ABCD}$ can now be expanded into a basis of Γ-matrices as follows:

$$B_m{}^A{}_B = \alpha_{m\,ab}\,\Gamma^{ab}_{AB} - i\,\alpha_{m\,a}\,\Gamma^a_{AB} + \alpha_{m\,abc}\,\Gamma^{abc}_{AB}, \tag{3.5}$$

$$A_{m\,ABCD} = \beta_{m\,ab}\,\Gamma^a_{[AB}\Gamma^b_{CD]} + i\,\beta_{m\,a}\,\Gamma^{ab}_{[AB}\Gamma^b_{CD]} + \beta_{m\,abc}\,\Gamma^{[a}_{[AB}\Gamma^{bc]}_{CD]}, \tag{3.6}$$

where the expansion coefficients are (anti-)symmetric according to the contraction with the Γ-matrices, but otherwise real and arbitrary. Substituting those expansions into (3.4), we obtain an equation that is antisymmetric in the spinor indices. The contraction with Γ^a_{AB} and Γ^{ab}_{AB} projects out two independent equations for the expansion coefficients:

$$\alpha_{m\,ab} + \tfrac{2}{3}\beta_{m\,ab} + \tfrac{1}{3}\delta_{ab}\beta_{m\,cc} = 0, \tag{3.7}$$

$$\alpha_{m\,abc} + \tfrac{2}{3}\beta_{m\,abc} - \tfrac{i}{3}\delta_{a[b}\left(\alpha_{m\,c]} + 4\beta_{m\,c]}\right) = 0, \tag{3.8}$$

where the antisymmetrization is only over the flat indices. From the symmetric and antisymmetric parts of (3.7) and the real and imaginary parts of (3.8), we obtain the general solution:

$$\alpha_{m\,ab} = 0, \qquad \beta_{m\,ab} = 0, \tag{3.9}$$

$$\alpha_{m\,a} + 4\beta_{m\,a} = 0, \qquad \alpha_{m\,abc} + \tfrac{2}{3}\beta_{m\,abc} = 0. \tag{3.10}$$

---

[9]Which from the $d{=}7$ perspective looks like a *rigid* transformation.

Rotating back with Killing spinors gives the general solution to (3.3), which may be partially recast in terms of the Killing vectors:

$$
\begin{aligned}
B_m{}^I{}_J &= -i\left(m_7\,\mathring{e}_{ma} + \alpha_{m\,a}\right)\bar{\eta}^I\Gamma^a\eta^J + \alpha_{m\,abc}\,\bar{\eta}^I\Gamma^{abc}\eta^J \\
&= -\left(m_7\,\delta_m{}^n + \alpha_m{}^n\right)K_n{}^{IJ} + \alpha_{m\,abc}\,\bar{\eta}^I\Gamma^{abc}\eta^J\,,
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
A_{m\,IJKL} &= i\,\beta_{m\,a}\,\bar{\eta}^{[I}\Gamma^{ab}\eta^J\bar{\eta}^K\Gamma^b\eta^{L]} + \beta_{m\,abc}\,\bar{\eta}^{[I}\Gamma^{ab}\eta^J\bar{\eta}^K\Gamma^c\eta^{L]} \\
&= -\beta_m{}^n\,\mathring{D}_n K_p^{[IJ}K^{p\,IJ]} + \beta_{m\,abc}\,\bar{\eta}^{[I}\Gamma^{ab}\eta^J\bar{\eta}^K\Gamma^c\eta^{L]}\,,
\end{aligned}
\tag{3.12}
$$

where

$$
\alpha_m{}^n = \alpha_{m\,a}\mathring{e}^{an}\,, \qquad \beta_m{}^n = \beta_{m\,a}\mathring{e}^{an}\,.
\tag{3.13}
$$

Finally, by applying the $E_{7(7)}$ transformation (3.2) to (3.11) and (3.12), we obtain the general solution to the GVP (2.14) on the $d=4$ side.

We see that the one parameter family in the standard solution (2.19) and (2.20) of the vielbein equation corresponds to the special choice

$$
\alpha_m{}^n = (\alpha - 1)\,m_7\,\delta_m{}^n\,, \qquad \beta_m{}^n = \beta\,\delta_m{}^n\,, \qquad \alpha + 4\beta = 1\,.
\tag{3.14}
$$

This family is distinguished in that $\mathcal{B}_m{}^i{}_j$ and $\mathcal{A}_{m\,ijkl}$ are constructed entirely from the 56-bein and the Killing vectors, and are 'covariant' with respect to the round S^7 in the sense that $\alpha_{m\,a} \sim \beta_{m\,a} \sim \mathring{e}_{ma}$. However, we will see in the following that the consistent truncation calls for more general solutions than the standard one.

To enumerate all solutions to the GVP in $d=4$, we introduce the independent parameters with flat indices,

$$
\alpha_{a|b} = \mathring{e}_a{}^m\,\alpha_{mb}\,, \qquad \alpha_{a|bcd} = \mathring{e}_a{}^m\,\alpha_{m\,bcd}\,,
\tag{3.15}
$$

which, as suggested by the notation, have no a priori symmetry between the first index and the remaining ones. Hence, under the SO(7) tangent rotations acting on the *background* siebenbein $\mathring{e}^a$, they decompose into the following irreducible components:

$$
\begin{aligned}
\alpha_{a|b} &= \alpha_{ab}^{\boxplus} + \alpha_{ab}^{\boxminus} + (\alpha - 1)m_7\,\delta_{ab}\,, \\
\alpha_{a|bcd} &= \alpha_{abcd}^{\boxvert} + \alpha_{abcd}^{\boxright} + \delta_{a[b}\tilde{\alpha}_{cd]}^{\boxminus}\,,
\end{aligned}
\tag{3.16}
$$

all of which will in general be present in any particular solution (note that α_{ab} and $\tilde{\alpha}_{ab}$ are different).

Let us now turn to the GVP (2.10) on the $d=11$ side. Using covariance of the GVP, the SU(8) gauge field $\mathcal{B}_m{}^A{}_B$ and the self-dual tensor $\mathcal{A}_{m\,ABCD}$ of the general solution in $d=11$ can be obtained by applying the SU(8) gauge transformation (the U-rotation) (2.23) to the general

solution for $\mathcal{B}_m{}^i{}_j$ and $\mathcal{A}_{m\,ijkl}$ in (3.11) and (3.12), respectively. However, since the U-rotation is not known explicitly, we will proceed differently and solve (2.10) directly on the $d\!=\!11$ side.

First we note that the decomposition of $\mathcal{B}_m{}^A{}_B$ and $\mathcal{A}_{m\,ABCD}$ into irreducible SU(8) components is given by precisely the same expansions as in (3.5) and (3.6), respectively, although values of the expansion parameters for a given solution will in general be different on the $d\!=\!4$ and the $d=11$ side. Next we evaluate the derivative of the generalized vielbein in (2.10), which, using (2.1) and (2.2), can be expressed in terms of the matrix $S^{-1}\mathring{D}_m S$ [4]. It is then straightforward to check that the first terms in (2.11) and (2.12) given by the antisymmetric and symmetric parts of $(S^{-1}\mathring{D}_m S)_{ab}$, respectively, already solve (2.10) by themselves and that the resulting homogeneous equation that determines parameters of the general solution is exactly the same as (3.4) or, equivalently, (3.9) and (3.10). Hence, we may readily write down the most general solution to (2.10) which, in accordance with (2.13), is given by

$$\mathcal{B}_m{}^A{}_B = \tfrac{1}{2}\,(S^{-1}\mathring{D}_m S)_{ab}\Gamma^a_{AB} + 4i\,X_{m\,a}\,\Gamma^a_{AB} - \tfrac{2}{3}\,X_{m\,bcd}\,\Gamma^{bcd}_{AB}\,, \qquad (3.17)$$

$$\mathcal{A}_{m\,ABCD} = -\tfrac{3}{4}\,(S^{-1}\mathring{D}_m S)_{ab}\Gamma^a_{[AB}\Gamma^b_{CD]} + i\,X_{m\,a}\,\Gamma^b_{[AB}\Gamma^{ab}_{CD]} + X_{m\,bcd}\,\Gamma^b_{[AB}\Gamma^{cd}_{CD]}\,, \qquad (3.18)$$

where $X_{m\,a}$ and $X_{m\,abc} = X_{m\,[abc]}$ are real and otherwise arbitrary.[10]

After conversion to flat indices, the tensors

$$X_{a|b} = e_a{}^m\,X_{m\,b}\,, \qquad X_{a|bcd} = e_a{}^m\,X_{m\,bcd}\,, \qquad (3.19)$$

may be decomposed into irreducible components under the SO(7) rotations acting on the metric vielbein e^a,

$$X_{a|b} = X_{ab}^{\square} + X_{ab}^{\boxminus} + \delta_{ab}\,X\,,$$

$$X_{a|bcd} = X_{abcd}^{\boxminus} + X_{abcd}^{\boxdown} + \delta_{a[b}\tilde{X}_{cd]}^{\boxminus}\,. \qquad (3.20)$$

with the same representation content as in (3.16).

Comparing with the solution (2.11) and (2.12), we see that the only SO(7) representations in (3.19) and (3.16) that are consistent with the supersymmetry in $d\!=\!11$ are the singlet and the totally antisymmetric one. Those are determined by the components of the flux,

$$X = \tfrac{\sqrt{2}}{56}\,f\,, \qquad X_{abcd}^{\boxminus} = \tfrac{\sqrt{2}}{32}F_{abcd}\,. \qquad (3.21)$$

The correct tensor structure condition simply means that the X-parameters in all other representations must vanish.

---

[10]We also rescaled them with respect to (2.13).

GV	$\mathrm{E}_{7(7)}$ connection	Parameters	GVP	Rotation
$i\,\Gamma^m_{AB}$	$B_m{}^A{}_B\,,\;A_{m\,ABCD}$	$\alpha_{a|b}\,,\alpha_{a|bcd}$	(3.4)	
K^m_{IJ}	$B_m{}^I{}_J\,,\;A_{m\,IJKL}$	$\alpha_{a|b}\,,\alpha_{a|bcd}$	(3.3)	$\eta \in \mathrm{SO}(8)$
e^m_{ij}	$\mathcal{B}_m{}^i{}_j\,,\;\mathcal{A}_{m\,ijkl}$	$\alpha_{a|b}\,,\alpha_{a|bcd}$	(2.14)	$\mathcal{V} \in \mathrm{E}_{7(7)}$
e^m_{AB}	$\mathcal{B}_m{}^A{}_B\,,\;\mathcal{A}_{m\,ABCD}$	$X_{a|b}\,,X_{a|bcd}$	(2.10)	$U \in \mathrm{SU}(8)$

Table 1: Generalized Vielbein Postulate in different frames.

In Table 1 we have summarized the different forms of the vielbein equation that we looked at in this section and listed the functions that parametrize the space of solutions. While it is clear that any solution on the $d=4$ side given in terms of $\alpha_{a|b}(x,y)$ and $\alpha_{a|bcd}(x,y)$ maps under the SU(8) rotation, $U(x,y)$, onto a unique solution given in terms of $S(x,y)$, $X_{a|b}(x,y)$ and $X_{a|bcd}(x,y)$, it is by no means guaranteed that the latter will be consistent with the tensor structure (3.21) required by $d=11$ supergravity.

4 The $\mathfrak{A}$-equations

To resolve the possible remaining discrepancies we need to take a closer look at $\mathfrak{A}$-equations (2.17) and (2.18) which, together with the GVP (2.14), guarantee the consistent reduction of the supersymmetry transformations from $d=11$ to $d=4$ [3]. The new key insight of the present work is that these equations admit more general solutions than the one given in [3], and encapsulated in the standard inhomogeneous solution (2.19)-(2.21).

We first show that the system of vielbein equations and $\mathfrak{A}$-equations is invariant under the action of $\mathrm{E}_{7(7)}$. To this end we recall that the A_1^{ij} and $A_{2\,l}{}^{ijk}$ tensors and their complex conjugates correspond to two SU(8) irreducible components of the so-called T-tensor of $\mathcal{N}=8$, $d=4$ gauged supergravity. The latter is defined in terms of the scalar 56-bein [1]

$$T_i{}^{jkl} = (u^{kl}{}_{IJ} + v^{klIJ})(u_{im}{}^{JK} u^{jm}{}_{KI} - v_{imJK} v^{jmKI})\,. \tag{4.1}$$

Then

$$A_1^{ij} = \tfrac{4}{21}\,T_k{}^{ikj}\,, \qquad A_{2\,l}{}^{ijk} = -\tfrac{4}{3}\,T_l{}^{[ijk]}\,. \tag{4.2}$$

It follows from the variations of the T-tensor in [1] that under infinitesimal transformations of the scalar vielbein,

$$\delta\mathcal{V} = \begin{pmatrix} \delta_{[i}{}^{[k}\Lambda_{j]}{}^{l]} & \Sigma_{ijkl} \\ \Sigma^{ijkl} & \delta^{[i}{}_{[k}\Lambda^{j]}{}_{l]} \end{pmatrix}\mathcal{V}\,, \tag{4.3}$$

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where $\Lambda^i{}_j$ are antihermitian and Σ_{ijkl} are self-dual, the A_1^{ij} and $A_{2\,i}{}^{ijkl}$ tensors together transform in the **912** irreducible representation of $E_{7(7)}$ [39]. [11] We will now show that the $E_{7(7)}$ transformations of the generalized vielbein and the tensor fields in (3.1) and (3.2) induce exactly the same $E_{7(7)}$ transformations of the composite $\mathfrak{A}_1^{ij}$ and $\mathfrak{A}_{2\,i}{}^{jkl}$ tensors defined in (2.15) and (2.16).

Since the SU(8) covariance is manifest, all we must show is that under infinitesimal transformation by the coset generators [39]

$$\delta\mathfrak{A}_1^{ij} = -\tfrac{1}{6}\left(\mathfrak{A}_2{}^i{}_{pqr}\Sigma^{jpqr} + \mathfrak{A}_2{}^j{}_{pqr}\Sigma^{ipqr}\right), \tag{4.4}$$

$$\delta\mathfrak{A}_{2\,i}{}^{jkl} = -2\,\mathfrak{A}_{1ip}\,\Sigma^{pjkl} - 3\,\Sigma^{pq[jk}\mathfrak{A}_2{}^{l]}{}_{ipq} - \Sigma^{pqr[j}\,\delta^k{}_i\,\mathfrak{A}_2{}^{l]}{}_{pqr}. \tag{4.5}$$

Evaluating $\delta\mathfrak{A}_1^{ij}$ from the definition (2.15) and setting it equal to (4.4) gives

$$\Sigma^{ijkl}\,\mathcal{B}_m{}^p{}_{[k}\,e^m_{l]p} + \Sigma_{klpq}\,\mathcal{A}_m{}^{ijpq}\,e^{m\,kl} + \tfrac{2}{3}\Sigma_{klpq}\,\mathcal{A}_m{}^{jklp}\,e^{m\,iq} + \tfrac{2}{3}\Sigma^{iklp}\,\mathcal{A}_{m\,klpq}\,e^{m\,jq} \overset{?}{=} 0. \tag{4.6}$$

Then using selfduality of Σ_{ijkl} and $\mathcal{A}_{m\,ijkl}$, we can rewrite the second and the third terms as

$$\Sigma_{klpq}\,\mathcal{A}_m{}^{ijpq}\,e^{m\,kl} = \Sigma^{ijkl}\,\mathcal{A}_{m\,klpq}\,e^{m\,pq} + \tfrac{1}{6}\Sigma^{klpq}\,\mathcal{A}_{m\,klpq}\,e^{m\,ij}$$
$$- \tfrac{2}{3}\left(\Sigma^{iklp}\,\mathcal{A}_{m\,klpq}\,e^{m\,jq} - \Sigma^{jklp}\,\mathcal{A}_{m\,klpq}\,e^{m\,iq}\right), \tag{4.7}$$

and

$$\tfrac{2}{3}\Sigma_{klpq}\,\mathcal{A}_m{}^{jklp}\,e^{m\,iq} = -\tfrac{1}{6}\Sigma^{klpq}\,\mathcal{A}_{m\,klpq}\,e^{m\,ij} - \tfrac{2}{3}\Sigma^{jklp}\,\mathcal{A}_{m\,klpq}\,e^{m\,iq}, \tag{4.8}$$

respectively. This reduces (4.6) to the vielbein equation,

$$\Sigma^{ijkl}\left(\mathcal{B}_m{}^p{}_{[k}\,e^m_{l]p} + \mathcal{A}_{m\,klpq}\,e^{m\,pq}\right) = 0, \tag{4.9}$$

where we have used that $\mathring{D}_m e^m_{ij} = 0$, which follows from the definition of the generalized vielbein in terms of the Killing vectors on S^7.

One can also check the $\delta\mathfrak{A}_1^{ij}$ equation (4.4) starting from the equivalent definition [3]

$$\mathfrak{A}_1^{ij} = \tfrac{\sqrt{2}}{4}\,e^{m\,k(i}\mathcal{B}_m{}^{j)}{}_k, \tag{4.10}$$

which makes the symmetry in (ij) manifest. Here, a simple substitution of variations yields the condition

$$\Sigma^{ipqr}\,\mathcal{A}_{m\,pqrs}\,e^{m\,js} + (i \leftrightarrow j) \overset{?}{=} 0. \tag{4.11}$$

---

[11]The modern formulation of gauged supergravities relies on the *embedding tensor formalism* [40, 41]. The above transformation property of the T-tensor then simply expresses the so-called *representation constraint* that the embedding tensor must satisfy.

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Denote $U^j{}_{pqr} = \mathcal{A}_{m\,pqrs}\,e^{mjs}$ and use the self-duality of the $E_{7(7)}$ generator to rewrite the left hand side in (4.10) as

$$\tfrac{1}{24}\,\Sigma_{xywz}\,\epsilon^{xywzipqr}\,U^j{}_{pqr} + (i \leftrightarrow j) = \tfrac{1}{24}\,\Sigma_{xywz}\,\epsilon^{ixywzpqr}\,\delta^s{}_p\,U^j{}_{sqr} + (i \leftrightarrow j)\,. \tag{4.12}$$

The vanishing of this expression follows now from the Schouten identity applied to the indices $xywzpqrsj$ and the explicit symmetry in (ij).

The $\delta\mathfrak{A}_{2i}{}^{jkl}$ equation obtained by comparing the variation of (2.16) with (4.5), is satisifed after using the vielbein equation and the selfduality of Σ_{ijkl} and $\mathcal{A}_{m\,ijkl}$. The intermediate expressions are more involved and we omit them here.

The $E_{7(7)}$ invariance of the $\mathfrak{A}$-equations (2.17) and (2.18) on the space of solutions of the generalized vielbein equation (2.14) allows us to solve those equations at the origin of the coset, where

$$A_1^{ij}\big|_{\mathcal{V}=\mathbf{1}} = \delta_{ij}\,, \qquad A_{2i}{}^{jkl}\big|_{\mathcal{V}=\mathbf{1}} = 0\,, \tag{4.13}$$

and $\mathcal{B}_m{}^i{}_j$ and $\mathcal{A}_{m\,ijkl}$ are given in (3.11) and (3.12). Furthermore, since the Killing spinors form an SO(8) matrix, they can be rotated away from all equations, which then involve only the parameters $\alpha_{a|b}$ and $\alpha_{a|bcd}$, and the Γ-matrices.

Let us first look at the $\mathfrak{A}_1$ equation (2.17). Using (3.11), (4.10) and (4.13) it becomes

$$g\,\delta_{ij} = \tfrac{\sqrt{2}}{4}\,i\,\Gamma^a_{k(i}\Big[-i\,(m_7\,\delta_{ab} + \alpha_{a|b})\Gamma^b_{j)k} + \alpha_{a|bcd}\,\Gamma^{bcd}_{j)k}\Big]\,. \tag{4.14}$$

Collecting the independent terms and using (2.22), we get

$$(3\,m_7 + \alpha_{a|a})\,\delta_{ij} - i\,\alpha_{a|bcd}\,\Gamma^{abcd}_{ij} = 0\,, \tag{4.15}$$

which sets the completely antisymmetric component $\alpha_{[a|bcd]}$ to zero and fixes the trace of $\alpha_{a|b}$ via (3.14) such that $\alpha = \tfrac{4}{7}$. All the other components of $\alpha_{a|b}$ and $\alpha_{a|bcd}$ are left arbitrary.

It is more tedious to check that the $\mathfrak{A}_2$-equation (2.18), which now simply reads

$$\mathfrak{A}_{2i}{}^{jkl}\big|_{\mathcal{V}=\mathbf{1}} = 0\,, \tag{4.16}$$

is also solved if (4.15) is satisfied. To check this explicitly, we note that (4.16) is antisymmetric in $[jkl]$, hence we may instead show that the two equations obtained by contracting (4.16) with Γ^a_{jk} and Γ^{ab}_{jk} are satisfied. This is actually easier than working with the original equation, which is a fourth-rank tensor that must be expanded in the basis of independent products of Γ-matrices. Instead, after the contraction, one ends up with a second rank tensor, which is much simpler. Still given all anti-symmetrizations in (2.16) and in (3.12), one ends up with a large number of terms which are best handled by a computer.

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To summarize, we have shown that a general solution to the generalized vielbein equation and the $\mathfrak{A}$-equations is given by

$$\alpha_{a|b} = -\tfrac{3}{7}\, m_7\, \delta_{ab} + \alpha_{ab}^{\,\square\square} + \alpha_{ab}^{\,\square}\,, \qquad \alpha_{a|bcd} = \alpha_{abcd}^{\,\square} + \delta_{a[b}\,\tilde{\alpha}_{cd]}^{\,\square}\,, \tag{4.17}$$

where we used the same notation as in (3.16). All parameters in (4.17) are completely arbitrary. The standard solution (2.19)-(2.21) is obtained by setting all those parameters to zero. Then from (3.14) we get $\alpha = \frac{4}{7}$ which agrees with (2.21).

The reader might have noticed that the SO(7) representations that arise in the solution (4.17) are precisely the same representations that must be set to zero in the parameters (3.19) of the general solution (3.17) and (3.18) to the GVP in $d=11$ to obtain standard form with the fluxes in (3.21). However, one must be careful here because, as we have noted at the end of Section 3, the relation between the parameters $\alpha_{a|b}$ and $\alpha_{a|bcd}$ on the $d=4$ side and the parameters $X_{a|b}$ and $X_{a|bcd}$ on the $d=11$ side is by no means straightforward as it involves, see Table 1, both the $E_{7(7)}$ and the SU(8) rotations that can mix different SO(7) components.

Let us now choose a particular solution in $d = 4$, for example the standard solution, $(\mathring{\mathcal{B}}_m{}^i{}_j, \mathring{\mathcal{A}}_{ijkl})$. Any other solution is then obtained by adding to it a solution, $(\delta\mathcal{B}_m{}^i{}_j, \delta\mathcal{A}_{m\,ijkl})$, to the homogenous equations given by setting the first term in (2.14) and the left hand sides in (2.17) and (2.18) to zero. Since $\delta\mathcal{B}_m{}^i{}_j$ and $\delta\mathcal{A}_{m\,ijkl}$ transform homogeneously under SU(8), we may now apply the U rotation to those homogeneous equations to revert to the real vielbein gauge (2.1). After converting to flat indices using the metric vielbein, $e_a{}^m$, and dropping the warp factor, the homogeneous part of the GVP reduces to

$$\delta\mathcal{B}_a{}^C{}_{[A}\Gamma^b_{B]C} + \delta\mathcal{A}_{a\,ABCD}\Gamma^a_{CD} = 0\,, \tag{4.18}$$

while the conditions for $(\delta\mathcal{B}_a{}^A{}_B, \delta\mathcal{A}_{a\,ABCD})$ to be 'in the kernel of the $\mathfrak{A}$-equations' become

$$\Gamma^{aC(A}\,\delta\mathcal{B}_a{}^{B)}{}_C = 0\,, \tag{4.19}$$

$$3\,\Gamma^{a\,[AB}\,\delta\mathcal{B}_a{}^{C]}{}_D - 3\Gamma^a_{EF}\,\delta\mathcal{A}_a^{EF[AB}\delta^{C]}{}_D - 4\,\delta\mathcal{A}_a{}^{ABCE}\Gamma^a_{ED} = 0\,. \tag{4.20}$$

Those are equations of the type we have already encountered and solved above in (3.4), (4.14) and (4.16), so we may readily write the solution

$$e_a{}^m\delta\mathcal{B}_m{}^A{}_B = -4\,i\,\delta X_{a|b}\,\Gamma^b_{AB} - \tfrac{2}{3}\,\delta X_{a|bcd}\,\Gamma^{bcd}_{AB}\,, \tag{4.21}$$

$$e_a{}^m\delta\mathcal{A}_{mABCD} = i\,\delta X_{a|b}\,\Gamma^c_{[AB}\Gamma^{ac}_{CD]} + \delta X_{a|bcd}\,\Gamma^b_{[AB}\Gamma^{cd}_{CD]}\,, \tag{4.22}$$

where

$$\delta X_{a|a} = 0\,, \qquad \delta X_{[a|bcd]} = 0\,. \tag{4.23}$$

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While it is straightforward to verify the solution for (4.19), the proof for (4.20) is more involved. We proceed similarly as before by first substituting (4.21) and (4.22) into (4.20) and then contracting this equation with Γ^a_{AB} and Γ^{ab}_{AB}, respectively, to show that the resulting expressions indeed vanish if (4.23) is satisfied. Again we omit the lengthy intermediate expressions.

This proves that given any solution on the $d=4$ side, we can always correct it such as to obtain after the U-rotation a solution with the tensor structure consistent with the $d=11$ supersymmetry. Moreover, that solution is unique and hence determines the fluxes f and F_{abcd} in terms of the scalar vielbein of the $\mathcal{N}=8$, $d=4$ theory. Furthermore, the constraints (4.23) on the homogeneous correction are precisely such that both f and F_{abcd} do not get modified in the process, and hence, at a given point on the scalar coset, can be read off from *any* solution on the $d=4$ side, even if that solution may not be the one that has the correct tensor structure after it is U-rotated to $d=11$. This both completes the proof of the consistent truncation in [3] and shows that one can extract correct fluxes from the standard solution that was found there.

5   The fluxes

In this section, we will outline in a systematic way two methods for computing the fluxes, f and F_{mnpq}, in terms of $d=4$ quantities. The first method follows directly from the discussion above and has been essentially spelled out already. The second one requires some additional work to prove the flux formulae (2.27) and (2.28).

For a fixed scalar 56-bein (2.5) in $d=4$, the starting point for computing the corresponding field configuration in $d=11$ is the triplet,

$$e^m_{ij}, \qquad \mathcal{B}_m{}^i{}_j, \qquad \mathcal{A}_{m\,ijkl}, \tag{5.1}$$

where e^m_{ij} is the generalized vielbein defined in (2.4) and $(\mathcal{B}_m{}^i{}_j\,, \mathcal{A}_{m\,ijkl})$ is a solution to the GVP (2.14) and the $\mathfrak{A}$-equations (2.17) and (2.18). One can either use the standard inhomogeneous solution (2.19)-(2.21), or any other solution if that is more convenient. It follows from the general solution to the GVP in Section 3 that, for a given generalized vielbein, $\mathcal{B}_m{}^i{}_j$ and $\mathcal{A}_{m\,ijkl}$ completely determine each other. Hence either of them contains the full information about the fluxes. Here we choose to work with $\mathcal{A}_{m\,ijkl}$ for the simple reason that it is an SU(8) tensor.

From the generalized vielbein we determine the metric, g_{mn}, and the warp factor, Δ, which in general are already quite difficult to obtain in a closed analytic form.

Next we turn to the fluxes. In the first method, we calculate the metric vielbein, $e_m{}^a$, and solve (2.8) for the SU(8) rotation matrix, U, and then use the latter to rotate $\mathcal{A}_{m\,ijkl}$ to $d=11$ according to (2.23). The resulting $\mathcal{A}_{m\,ABCD}$ tensor is a solution to the GVP (2.10) and thus of the general form given in (3.18). We then read off the expansion coefficients, $X_{a|b}$ and $X_{a|bcd}$,

from which the fluxes are obtained by projecting onto irreducible SO(7) components, see (3.21),

$$f = 4\sqrt{2}\, X_{a|a}\,, \qquad F_{abcd} = 16\sqrt{2}\, X_{[a|bcd]}\,. \qquad (5.2)$$

As was shown above, the result does *not* depend on which particular solution $\mathcal{A}_{m\,ijkl}$ of the GVP we start with on the $d=4$ side.

A difficulty one encounters in trying to apply this construction in any specific example is that both the metric vielbein, $e_m{}^a$, and the SU(8) rotation, U, are obtained by solving quadratic equations, and can become quite complicated, if calculable analytically at all. Hence, one would like to avoid having to perform the rotation explicitly by working with SU(8) invariant quantities as in (2.9). The method how to do this was outlined in [3], and here we expand on it.

We start on the $d=11$ side with $\mathcal{A}_{m\,ABCD}$ that is obtained from $\mathcal{A}_{m\,ijkl}$ by the U-rotation (2.23). Then the fluxes are given by the projections (5.2) of the expansion coefficients of $\mathcal{A}_{m\,ABCD}$ in (3.18), so all that is needed is an effective way to extract those two projections from the $d=4$ result. The problem here is that the basis of the Γ-matrices used in (3.18) is not SU(8)-covariant and U-rotation mixes different terms in the expansion.

In order to construct SU(8)-covariant projections, we note that (2.1) and (2.8) imply a covariant transformation between the Γ-matrices on the $d=11$ side and the generalized vielbeine on the $d=4$ side, namely

$$\Gamma^a_{AB} U^A{}_i U^B{}_j = -i\,\Delta^{1/2}\, e_m{}^a\, e^m_{ij}\,, \qquad (5.3)$$

and

$$\Gamma^a_{AB} (U^A{}_i)^* (U^B{}_j)^* = -i\,\Delta^{1/2}\, e_m{}^a\, e^{m\,ij}\,, \qquad (5.4)$$

and it is this covariance that must be preserved. In particular, it implies that an SU(8)-covariant basis for the expansion of the $\mathcal{A}_{m\,ABCD}$ tensor must be constructed from odd products of Γ-matrices, which are then U-rotated into SU(8)-invariant contractions of the vielbeine e^m_{ij} and $e^{m\,ij}$. To implement this change of basis, we use Γ-matrix identities,

$$\Gamma^{ab}_{AB} = -\tfrac{i}{5!}\,\epsilon^{abcdefg}\Gamma^{cdefg}_{AB}\,, \qquad \Gamma^{abcde}_{AB} = \tfrac{i}{2}\,\epsilon_{abcdefg}\,\Gamma^{fg}_{AB}\,, \qquad (5.5)$$

and

$$\Gamma^a_{[AB}\Gamma^{bcdef}_{CD]} = \Gamma^{[a}_{[AB}\Gamma^{bcdef]}_{CD]} + \tfrac{5}{3}\,\delta^{a[b}\Gamma^{|g|}_{[AB}\Gamma^{cdef]g}_{CD]}\,, \qquad (5.6)$$

to recast (3.18) in the form

$$\mathcal{A}_{m\,ABCD} = -\tfrac{3}{4}\,(S^{-1}\mathring{D}_m S)_{ab}\Gamma^a_{[AB}\Gamma^b_{CD]} + i\,X_{m\,abcd}\,\Gamma^f_{[AB}\Gamma^{abcdf}_{CD]} + X_{m\,abcdef}\,\Gamma^{[a}_{[AB}\Gamma^{bcdef]}_{CD]}\,, \qquad (5.7)$$

where $X_{m\,abcd}$ and $X_{m\,abcdef}$ are completely antisymmetric in their (flat) indices, and are related to the original expansion coefficients in (3.18) by

$$X_{m\,abcd} = -\tfrac{1}{3\cdot 4!}\epsilon_{abcdefg}\,X_{m\,efg}\,, \qquad (5.8)$$

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$$X_{m\,abcdef} = \tfrac{1}{5!}\epsilon_{abcdefg}\,X_{m\,g}\,. \tag{5.9}$$

Then from (5.2) we obtain[12]

$$f = 4\sqrt{2}\,e_a{}^m X_{ma} = \tfrac{2\sqrt{2}}{3}\,\epsilon^{abcdefg}\,e_a{}^m\,X_{m\,bcdefg}\,, \tag{5.10}$$

and

$$F_{abcd} = -8\sqrt{2}\,e_{[a}{}^m\epsilon_{bcd]}{}^{efgh}X_{m\,efgh}\,. \tag{5.11}$$

One should note the antisymmetrization in (5.11) that projects onto the correct tensor structure of the four-form flux in $d=11$ supergravity.

To project out the components (5.8) and (5.9) from the $\mathcal{A}_{m\,ABCD}$ tensor, we can simply contract with the basis tensors $\Gamma^f_{[AB}\Gamma^{abcdf]}_{CD]}$ and $\Gamma^{[a}_{[AB}\Gamma^{bcdef]}_{CD]}$, which are orthogonal with respect to each other and normalized according to

$$\Gamma^c_{[AB}\Gamma^{a_1a_2a_3a_4c}_{CD]}\,\Gamma^d_{AB}\Gamma^{b_1b_2b_3b_4d}_{CD} = 48\cdot 4!\,\delta^{a_1a_2a_3a_4}_{b_1b_2b_3b_4}\,, \tag{5.12}$$

$$\Gamma^{[a_1}_{[AB}\Gamma^{a_2a_3a_4a_5a_6]}_{CD]}\Gamma^{[b_1}_{AB}\Gamma^{b_2b_3b_4b_5b_6]}_{CD} = 32\cdot 5!\,\delta^{a_1a_2a_3a_4a_5a_6}_{b_1b_2b_3b_4b_5b_6}\,. \tag{5.13}$$

The same projections in terms of SU(8)-covariant products of the vielbeine, e^m_{ij} and $e^{m\,ij}$, obtained using (5.3) and (5.4), can be applied to $\mathcal{A}_{m\,ijkl}$. In this way, after passing to curved indices, we obtain explicit formulae for the fluxes expressed entirely in terms of $d=4$ quantities:

$$f = -\tfrac{\sqrt{2}}{48\cdot 5!}\,\Delta^4\,g^{mu}\,\epsilon_{mnpqrst}\,e^n_{ij}\,(e^{[p}\bar{e}^q e^r \bar{e}^s e^{t]})_{kl}\,\mathcal{A}_u{}^{ijkl}\,, \tag{5.14}$$

and

$$F_{mnpq} = -\tfrac{i}{144}\,\Delta^4\,g_{rw}\,e^r_{ij}(e^{[s}\bar{e}^t e^u \bar{e}^v e^{w]})_{kl}\,\epsilon_{stuv[mnp}\,\mathcal{A}_{q]}{}^{ijkl}\,. \tag{5.15}$$

Those expressions for the fluxes are valid for any $\mathcal{A}_{m\,ijkl}$ satisfying the GVP and the $\mathfrak{A}$-equations. In particular, they hold for $\mathring{\mathcal{A}}_{m\,ijkl}$, the standard solution (2.20).

For the particular $\mathcal{A}_{m\,ijkl}$ that has the correct tensor structure in $d=11$, we may combine (5.14) and (5.15) into (2.26), however, for other $\mathcal{A}_{m\,ijkl}$ this equation will in general not hold if there are other than just the flux components in $\mathcal{A}_{m\,ABCD}$. We will illustrate this on some examples in the next section.

Finally, let us note that by plugging in the general solution for $\mathcal{A}_{m\,ijkl}$ derived in Sections 3 and 4, and parametrized in terms of $\alpha_{a|b}$ and $\alpha_{a|bcd}$, in the flux formulae (5.14) and (5.15), we can directly relate two of the X-parameters, namely, X and $X^{\boxed{}}_{abcd}$, to the α-parameters. Working out projection formulae for the other SO(7) irreducible components, similar relations can be obtained for other parameters as well. However, it is clear that, largely because of the $E_{7(7)}$ rotation that mixes different Γ-matrix structures, the final formulae will be quite involved and not very illuminating.

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[12]The indices on the Levi-Civita symbol are raised and lowered with the background metric.

6 Analytic examples

In this section we illustrate various points of the general discussion on three examples: (i) the SO(8) critical point, (ii) the SO(7)$^-$ invariant family, and (iii) the SO(7)$^+$ invariant family, for which the U-rotations are known in a closed form.

6.1 SO(8)

We begin with the maximally supersymmetric critical point,

$$u_{ij}{}^{IJ} = \delta_{ij}{}^{IJ}, \qquad v_{ijIJ} = 0, \tag{6.1}$$

corresponding to the $AdS_4 \times S^7$ solution [23] of $d=11$ supergravity,

$$e_m{}^a = \mathring{e}_m{}^a, \qquad f = 3\sqrt{2}\, m_7, \qquad F_{abcd} = 0. \tag{6.2}$$

The generalized vielbein and the U-rotation are simply,

$$e_{ij}^m = i\, \mathring{e}_a{}^m\, \bar{\eta}^i \Gamma^a \eta^j, \qquad U^i{}_A = \eta^i{}_A, \tag{6.3}$$

and the general solution to the GVP and the $\mathfrak{A}$-equations, c.f. (4.17), is

$$\mathring{e}_a{}^m\, A_{m\,ijkl} = -\tfrac{i}{4}\left(-\tfrac{3}{7}\, m_7\, \delta_{ab} + \alpha_{ab}^{\square} + \alpha_{ab}^{\boxplus}\right) \bar{\eta}_{[i}\Gamma^{bc}\eta_j\bar{\eta}_k\Gamma^c\eta_{l]} - \tfrac{3}{2}\left(\alpha_{abcd}^{\boxplus} + \delta_{a[b}\tilde{\alpha}_{cd]}^{\boxplus}\right)\bar{\eta}_{[i}\Gamma^{[bc}\eta_j\bar{\eta}_k\Gamma^{d]}\eta_{l]}. \tag{6.4}$$

Since the U-rotation is merely a change between the two types of SU(8) indices, the X-parameters are proportional to the α-parameters,

$$X_{a|b} = -\tfrac{1}{4}\,\alpha_{a|b}, \qquad \alpha_{a|bcd} = -\tfrac{2}{3}\, X_{a|bcd}, \tag{6.5}$$

and, in particular, we have

$$X = \tfrac{3}{4}\, m_7, \qquad X_{abcd}^{\boxplus} = 0, \tag{6.6}$$

from which the fluxes (6.2) follow.

We also note that by setting the explicit α-parameters in (6.4) to zero, we recover the standard inhomogenous solution, which in this example satisfies the correct tensor structure condition. This is not surprising, as by construction the standard inhomogenous solution (2.19) and (2.20) has the same symmetry as the scalar background in $d = 4$, and the only SO(8)-invariant α-parameter that one can have is a constant singlet. The formula for the fluxes based on the standard solution was already tested in [3].

We may also verify the fluxes by evaluating the projections (5.14) and (5.15). From the orthonormality of the Killing spinors and Γ-matrix identities, we have

$$\epsilon_{mnpqrst}\, e_{ij}^n\, (e^{[p}\bar{e}^q e^r \bar{e}^s e^{t]})_{kl}\, \bar{\eta}_{[i}\Gamma^{bc}\eta_j\bar{\eta}_k\Gamma^c\eta_{l]} = 192 \cdot 5!\, i\, \mathring{e}_m{}^a, \tag{6.7}$$

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and

$$\epsilon_{mnpqrst}\, e^n_{ij}\, (e^{[p}\bar{e}^q e^r \bar{e}^s e^{t]})_{kl}\, \bar{\eta}_{[i}\Gamma^{[bc}\eta_j \bar{\eta}_k \Gamma^{d]}\eta_{l]} = 0\,, \tag{6.8}$$

where (6.8) also follows from the SO(8) invariance. Substituting those contractions in (5.14) and using the tracelessness of $\alpha^{\Box}_{ab}$ and α^{θ}_{ab}, we get

$$f = -\left(\tfrac{\sqrt{2}}{48\cdot 5!}\right) \times \left(\tfrac{3}{28}i\right) \times (192\cdot 5!\, i) \times 7\, m_7 = 3\sqrt{2}\, m_7\,. \tag{6.9}$$

The vanishing of the internal flux is verified similarly.

6.2   SO(7)$^-$

The SO(7)$^-$-invariant sector of $\mathcal{N}=8$, $d=4$ supergravity provides the simplest example with a nontrivial U-rotation [42]. By symmetry, the standard inhomogenous solution must have the correct tensor structure in $d=11$ and we verify that at the SO(7)$^-$ critical point it yields the fluxes of Englert's 'parallelizing torsion solution' of $\mathcal{N}=1$, $d=11$ supergravity [43].

The scalar 56-bein in this sector forms a one parameter family [39]

$$u_{ij}{}^{IJ}(t) = u_1(t)\,\delta^{IJ}_{ij} + u_2(t)\,C^{ijIJ}_-\,, \qquad v_{ijIJ}(t) = v_1(t)\,\delta^{IJ}_{ij} + v_2(t)\,C^{ijIJ}_-\,, \tag{6.10}$$

where C^{IJKL}_- is an anti-selfdual tensor satisfying

$$C^{IJMN}_- C^{MNKL}_- = 12\delta^{IJ}_{KL} - 4C^{IJKL}_-\,, \tag{6.11}$$

and

$$\begin{aligned}
u_1(t) &= \cosh^3(2t)\,, & u_2(t) &= \tfrac{1}{2}\cosh(2t)\sinh^2(2t)\,, \\
v_1(t) &= i\sinh^3(2t)\,, & v_2(t) &= \tfrac{i}{2}\cosh^2(2t)\sinh(2t)\,.
\end{aligned} \tag{6.12}$$

The scalar potential along the SO(7)$^-$ family is

$$\mathcal{P}(t) = -2\, g^2 \cosh^5(4t)(5 - 2\cosh(8t))\,, \tag{6.13}$$

and has two critical points: the maximally supersymmetric one at $t=0$, and the SO(7)$^-$ point at $t=\tfrac{1}{4}\operatorname{arcoth}(\sqrt{5})$.

The solution of $d=11$ supergravity corresponding to the SO(7)$^-$ point can be expressed entirely in terms of an SO(7)$^-$ invariant rank three tensor, S_{abc}, on S^7,

$$S_{abc} = \frac{i}{16}\, C^{IJKL}_-\, \bar{\eta}^I\Gamma_{[ab}\eta^J\, \bar{\eta}^K\Gamma_{c]}\eta^L\,, \tag{6.14}$$

known as the 'parallelizing torsion,' in terms of which the generalized vielbein is given by

$$e^m_{ij} = i\,(u_1 + v_1)\,\bar{\eta}^i\mathring{\Gamma}^m\eta^j + (u_2 + v_2)\,S_{mab}\,\bar{\eta}^i\Gamma^{ab}\eta^j\,. \tag{6.15}$$

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As usual, the conversion between the flat/curved indices on the Γ-matrices and the S_{abc} tensor is done with the background vielbeins. To derive (6.15), one uses the 'inverse' of (6.14), see [42],

$$C_-^{IJKL} = \tfrac{i}{2} S_{abc} \, \bar{\eta}^{[I} \Gamma^{ab} \eta^J \bar{\eta}^K \Gamma^c \eta^{L]} \,. \tag{6.16}$$

Then, using[13]

$$S_{acd} S_{bcd} = 6 \, \delta_{ab} \,, \tag{6.17}$$

the metric and the warp factor are calculated from the metric lift formula (2.9):

$$\Delta^{-1} g^{mn} = \cosh^3(4t) \, \mathring{g}^{mn} \,, \qquad \Delta^{-1} = \cosh^{7/3}(4t) \,. \tag{6.18}$$

The metric vielbein, $e_m{}^a$, can be chosen to be proportional to $\mathring{e}_m{}^a$,

$$e_m{}^a = \cosh^{-1/3}(4t) \mathring{e}_m{}^a \,, \tag{6.19}$$

which means that the S-terms in $\mathcal{A}_{m\,ABCD}$ and $\mathcal{B}_m{}^A{}_B$ will be absent. As we already pointed out, once the normalization at the SO(8) point is fixed, the overall normalization of the $d = 11$ metric (as well as the other fields) is fixed along the whole family, and in particular at the SO(7)$^-$ stationary point.

The $U = \Phi \eta$ matrix for those vielbeine was calculated in [42],

$$\Phi = \tfrac{1}{8}(e^{-7i\tau} + 7e^{i\tau}) + \tfrac{i}{48}(e^{-7i\tau} - e^{i\tau}) S_{abc} \Gamma^{abc} \,, \tag{6.20}$$

where the parameter τ is related to t by

$$\tan(2\tau) = \tanh(2t) \,. \tag{6.21}$$

To evaluate $\mathring{\mathcal{A}}_{m\,ijkl}$ of the standard inhomogenous solution (2.20), we first rewrite the Killing vectors in terms of the Killing spinors (2.6). It follows from (2.6) and (2.7) that

$$\mathring{D}_m K_n{}^{IJ} = -m_7 \, \bar{\eta}^I \mathring{\Gamma}_{mn} \eta^J \,, \tag{6.22}$$

and we use it to similarly rewrite the second term in (2.20). Finally, after using (6.16) in the scalar 56-bein (6.10), and the orthonormality of the Kiling spinors, we are left with some tedious Γ-matrix algebra that is required to simplify the resulting expressions. A number of useful identities to do that can be found in [28] and [4]. The result is

$$\mathring{\mathcal{A}}_{m\,ijkl} = -\tfrac{i}{28}\, m_7 \sinh(8t) \mathring{e}_{md} \, S_{abc} \eta^{[i} \Gamma^{da} \eta^j \bar{\eta}^k \Gamma^{bc} \eta^{l]} + \tfrac{3i}{28}\, m_7 \cosh(4t) \mathring{e}_{mb} \bar{\eta}_{[i} \Gamma^{ba} \eta_j \bar{\eta}_k \Gamma^a \eta_{l]} \,. \tag{6.23}$$

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[13]For a full list of identities satisfied by torsion tensor, see [39]. They imply that any contraction and background derivative of the torsion tensor(s) can be reduced to terms that are linear in it.

The U-rotation to $d=11$ requires similar algebra, but is even more tedious and we omit here the intermediate steps. At the end we find

$$\mathring{\mathcal{A}}_{m\,ABCD} = \tfrac{i}{28}\,m_7(5 - 2\cosh(8t))\mathring{e}_m{}^a\Gamma^{ab}_{[AB}\Gamma^b_{CD]}$$
$$+ \tfrac{1}{48}\,m_7\sinh(4t)\mathring{e}_m{}^a\epsilon_{abcdefg}S_{efg}\Gamma^b_{[AB}\Gamma^{cd}_{CD]}\,. \tag{6.24}$$

We see that since $e_m{}^a$ and $\mathring{e}_m{}^a$ are proportional, this tensor has the correct tensor structure (2.12) and we readily read off the fluxes not just at the critical points, but along the entire $SO(7)^-$ family,

$$f = \sqrt{2}m_7\cosh^{1/3}(4t)(5 - 2\cosh(8t))\,, \tag{6.25}$$

$$F_{abcd} = \tfrac{\sqrt{2}}{3}\,m_7\cosh^{1/3}(4t)\sinh(4t)\,\epsilon_{abcdefg}S_{efg}\,. \tag{6.26}$$

Once more the correct tensor structure of (6.24) is guaranteed by the $SO(7)^-$ symmetry – one cannot construct from the torsion tensor, S_{abc}, and the vielbein, $e_m{}^a$, any other coefficients $X_{a|b}$ and $X_{a|bcd}$ than the singlet and the completely antisymmetric tensor, respectively. However, the normalization of each term at the $SO(7)^-$ critical point must agree with the known solution, which is a nontrivial test of our flux formulae.

The solution in [39, 42] is[14]

$$\widetilde{f} \equiv -\tfrac{i}{24}\,\Delta^{-1/2}\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta\gamma\delta} = \sqrt{2}\,\widetilde{m}_7\left(3 - 4\tan^2(4\tau)\right), \tag{6.27}$$

$$F_{abc} = \tfrac{1}{24}\,\Delta^{-1/2}\epsilon_{abc}{}^{defg}F_{defg} = 2\sqrt{2}\,\widetilde{m}_7\tan(4\tau)S_{abc}\,, \tag{6.28}$$

where the flux components, $F_{\alpha\beta\gamma\delta}$ and F_{abcd}, are with respect to the vielbein of the same form as in (6.19), while $\widetilde{f}$ is rescaled with respect to (1.2),

$$\widetilde{f} = \Delta^{-1/2}\,f\,. \tag{6.29}$$

Then, noticing that from (6.21) we have

$$5 - 2\cosh(8t) = 3 - 4\tan^2(4\tau)\,, \quad \sinh(4t) = \tan(4\tau)\,, \tag{6.30}$$

we get

$$f = \sqrt{2}\,\widetilde{m}_7\Delta^{1/2}\left(5 - 2\cosh(8t)\right), \tag{6.31}$$

$$F_{abcd} = \tfrac{1}{6}\Delta^{1/2}\epsilon_{abcd}{}^{efg}F_{efg} = \tfrac{\sqrt{2}}{3}\,\widetilde{m}_7\,\Delta^{1/2}\,\sinh(4t)\epsilon_{abcdefg}S_{efg}\,. \tag{6.32}$$

We see that those fluxes agree with the ones in (6.25) and (6.26), provided we set

$$\widetilde{m}_7 = \cosh^{3/2}(4t)\,m_7\,. \tag{6.33}$$

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[14]Since the same symbol in [39, 42] may denote quantities that are related by a rescaling to the ones here, we put a tilde whenever the identification is not immediately obvious.

There is no explicit expression for $\widetilde{m}_7$ as a functions of t in [42], except that its relation to the gauge coupling constant, g, changes along the family. Since we have $g = \sqrt{2}\,m_7$, we deduce from (6.33) that this relation must be

$$g = \sqrt{2}\,\cosh^{-3/2}(4t)\,\widetilde{m}_7\,. \tag{6.34}$$

This implies that at the maximally supersymmetric point,

$$g = \sqrt{2}\,\widetilde{m}_7\,, \tag{6.35}$$

while at the SO(7)$^-$ point,

$$g = 4 \cdot 5^{-3/4}\,\widetilde{m}_7\,. \tag{6.36}$$

This agrees with (1.4) in [42] and is our first new non-trivial test of the lift formulae for the fluxes.

6.3   SO(7)$^+$

The SO(7)$^+$ solution of $\mathcal{N}=1$, $d=11$ supergravity is constructed in terms of an invariant vector field ξ^a on S^7 [44]. This implies that the $X_{a|bcd}$ parameters of the standard inhomogenous solution in $d=11$ must vanish, which agrees with $F_{abcd} = 0$ [44], but allows for $X_{a|b}$ with both trace and traceless components. We will show that in fact the standard inhomogeneous solution has a non-vanishing $X_{ab}^{\boxplus}$ term, which can be removed by a suitable homogeneous correction, and that the resulting flux, f, agrees with the known solution. Throughout this subsecton we set $m_7 = 1$.

The scalar 56-bein of the one-parameter SO(7)$^+$ invariant family is

$$u_{ij}{}^{IJ}(t) = u_1(t)\,\delta_{ij}^{IJ} + u_2(t)\,C_+^{ijIJ}\,, \qquad v_{ijIJ}(t) = v_1(t)\,\delta_{ij}^{IJ} + v_2(t)\,C_+^{ijIJ}\,, \tag{6.37}$$

where C_+^{IJKL} is a self-dual tensor satisfying

$$C_+^{IJMN}C_+^{MNKL} = 12\,\delta_{KL}^{IJ} + 4\,C_+^{IJKL}\,, \tag{6.38}$$

and

$$\begin{aligned} u_1(t) &= \cosh^3(2t)\,, & u_2(t) &= \tfrac{1}{2}\cosh(2t)\sinh^2(2t)\,, \\ v_1(t) &= \sinh^3(2t)\,, & v_2(t) &= \tfrac{1}{2}\cosh^2(2t)\sinh(2t)\,. \end{aligned} \tag{6.39}$$

All the dependence on the internal geometry can be expressed in terms of the vector field,

$$\xi^a(y) = \tfrac{i}{16}\,C_+^{IJKL}\,\bar{\eta}^I\Gamma^{ab}\eta^J\,\bar{\eta}^K\Gamma^b\eta^L\,, \tag{6.40}$$

and a scalar function, $\xi(y)$, defined by

$$\xi^a\xi^a = (3 - \xi)(21 + \xi)\,. \tag{6.41}$$

A complete list of identities satisfied by ξ^a and ξ and their derivatives as well as further discussion of their properties can be found in [44].

Once more, (6.40) can be inverted by [44],

$$C_+^{IJKL} = \tfrac{1}{12}\,(9+\xi)\,\bar\eta^{[I}\Gamma^a\eta^J\bar\eta^K\Gamma^a\eta^{L]}$$
$$- \frac{\xi^a\xi^b}{4(3-\xi)}\,\bar\eta^{[I}\Gamma^a\eta^J\bar\eta^K\Gamma^a\eta^{L]} + \tfrac{i}{12}\,\xi^a\,\bar\eta^{[I}\Gamma^{ab}\eta^J\bar\eta^K\Gamma^b\eta^{L]}\,, \tag{6.42}$$

which is then used to rewrite the 56-bein (6.37). It is then straightforward to obtain the generalized vielbein (2.4), which reads

$$e_{ij}^m = \left[(u_1+v_1) - \tfrac{1}{3}(u_2+v_2)(3+\xi)\right]\,i\,\bar\eta^i\Gamma^m\eta^j$$
$$+ (u_2+v_2)\,\frac{\xi^m\,\xi^a}{3(3-\xi)}\,i\,\bar\eta^i\Gamma^a\eta^j + \tfrac{1}{3}(u_2+v_2)\,\xi^a\,\bar\eta^i\Gamma^{am}\eta^j\,, \tag{6.43}$$

where the conversion to curved indices is with the background vielbein, $\mathring{e}_a{}^m$.

At this point it is convenient to switch to the parameter,

$$\tau = \frac{e^{8t}-1}{3\left(e^{8t}+7\right)}\,, \tag{6.44}$$

introduced in [44], and absorb any dependence on ξ into the function

$$H(\xi,\tau) = \frac{(1+21\tau)}{\sqrt{(1+63\tau^2) - 2\,\xi\,\tau\,(1+9\tau)}}\,. \tag{6.45}$$

In the new parametrization, the maximally supersymmetric critical point is at $\tau = 0$, while the $SO(7)^+$ critical point is at $\tau = \tfrac{1}{33}\left(2\sqrt5 - 3\right)$ or, equivalently, at $16\,t = \ln 5$.

From the generalized vielbein (6.43), one calculates the metric, the warp factor and the metric vielbein. The latter is given by [26]

$$e_m{}^a = \lambda^{1/2}\left[\delta_m{}^a - \left(1 - \frac{1}{H}\right)\boldsymbol{\xi}_m\boldsymbol{\xi}^a\right]\,, \tag{6.46}$$

where $\boldsymbol{\xi}^a$ is the unit vector field corresponding to ξ^a, and

$$\lambda = \frac{(1-3\tau)^{1/3}}{(1+21\tau)^{1/3}}\,H^{2/3}\,. \tag{6.47}$$

The warp factor is

$$\Delta = \frac{(1-3\tau)^{7/6}}{(1+21\tau)^{7/6}}\,H^{4/3}\,. \tag{6.48}$$

We note that the overall τ-dependent normalization factors in (6.47) and (6.48), that follow from the lift formula for the metric (2.9), differ from those in [26] which were obtained by a

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different method. While this is irrelevant for a solution at the critical point, the correct τ-dependent normalization is needed to obtain lifts of more general solutions, such as RG-flows.

The $U = \Phi\eta$ matrix was given in [26] and it reads

$$\Phi = \cos\vartheta + \sin\vartheta\,(i\,\boldsymbol{\xi}^a\Gamma^a)\,, \tag{6.49}$$

where

$$\cos(2\vartheta) = \frac{H}{(1+21\tau)}(1-\tau\xi)\,, \qquad \sin(2\vartheta) = \frac{H\tau}{(1+21\tau)}\sqrt{(21+\xi)(3-\xi)}\,. \tag{6.50}$$

Note that in this example Φ (and U) is a real and hence an orthogonal matrix.

The evaluation of the $\mathring{\mathcal{A}}_{m\,ijkl}$ tensor (2.20) of the standard inhomogenous solution follows the same steps as in the $SO(7)^-$ example, but is much more tedious as may be inferred from the presence of higher rank symmetric tensors that can be constructed from products of ξ^a's. We refer the reader to [4, 44] for some useful identities.

Let us expand the tensor $\mathcal{A}_{m\,ijkl}(\alpha,\beta)$ on the left hand side in (2.20) as

$$\mathcal{A}_{m\,ijkl}(\alpha,\beta) = \alpha\,\mathcal{A}^{(\alpha)}_{m\,ijkl} + \beta\,\mathcal{A}^{(\beta)}_{m\,ijkl}\,. \tag{6.51}$$

The tensors on the right hand side, written in the form similar to (6.43) that allows us to trace the origin of individual terms, are given by

$$
\begin{aligned}
A^{(\alpha)}_{m\,ijkl} = (u_1 v_2 - u_2 v_1)\Big[&\tfrac{1}{2}\,\xi^a\,\bar{\eta}_{[i}\Gamma_m\eta_j\bar{\eta}_k\Gamma^a\eta_{l]} - \tfrac{1}{6}\,\xi_m\,\bar{\eta}_{[i}\Gamma^a\eta_j\bar{\eta}_k\Gamma^a\eta_{l]} \\
&- \tfrac{i}{12}\,(3-\xi)\,\bar{\eta}_{[i}\Gamma_m{}^a\eta_j\bar{\eta}_k\Gamma^a\eta_{l]} + \tfrac{i}{12}\,\frac{\xi_m\xi^a}{(3-\xi)}\,\bar{\eta}_{[i}\Gamma^{ab}\eta_j\bar{\eta}_k\Gamma^b\eta_{l]}\Big]\,,
\end{aligned}
\tag{6.52}
$$

$$
\begin{aligned}
A^{(\beta)}_{m\,ijkl} = &(u_1^2 - v_1^2)\,\bar{\eta}_{[i}\Gamma_m{}^a\eta_j\bar{\eta}_k\Gamma^a\eta_{l]} \\
&+ (u_1 u_2 - v_1 v_2)\Big[\tfrac{4i}{3}\,\xi^a\,\bar{\eta}_{[i}\Gamma_m\eta_j\bar{\eta}_k\Gamma^a\eta_{l]} \\
&\qquad\qquad - \tfrac{2}{9}\,(3+\xi)\,\bar{\eta}_{[i}\Gamma_m{}^a\eta_j\bar{\eta}_k\Gamma^a\eta_{l]} + \tfrac{2}{9}\,\frac{\xi_m\xi^a}{(3-\xi)}\,\bar{\eta}_{[i}\Gamma_m{}^a\eta_j\bar{\eta}_k\Gamma^a\eta_{l]}\Big] \\
&+ (u_2^2 - v_2^2)\Big[-\tfrac{16i}{3}\,\xi^a\,\bar{\eta}_{[i}\Gamma_m\eta_j\bar{\eta}_k\Gamma^a\eta_{l]} - \tfrac{2i}{9}\,(9+\xi)\,\xi_m\,\bar{\eta}_{[i}\Gamma^a\eta_j\bar{\eta}_k\Gamma^a\eta_{l]} \\
&\qquad\qquad + \tfrac{2i}{3}\,\frac{\xi_m\xi^a\xi^b}{(3-\xi)}\,\bar{\eta}_{[i}\Gamma^a\eta_j\bar{\eta}_k\Gamma^b\eta_{l]} + \tfrac{4}{9}(15+2\xi)\,\bar{\eta}_{[i}\Gamma_m{}^a\eta_j\bar{\eta}_k\Gamma^a\eta_{l]} \\
&\qquad\qquad - \tfrac{2}{9}\,\frac{\xi_m\xi^a}{(3-\xi)}\,\bar{\eta}_{[i}\Gamma^{ab}\eta_j\bar{\eta}_k\Gamma^b\eta_{l]}\Big]\,.
\end{aligned}
\tag{6.53}
$$

The U-rotation is now effectively a substitution in (6.52) and (6.53) of the form

$$\bar{\eta}_i\Gamma^a\eta_j \;\longmapsto\; \cos(2\vartheta)\,\Gamma^a_{AB} - i\,\sin(2\vartheta)\,\boldsymbol{\xi}^b\,\Gamma^{ab}_{AB} + 2\sin^2\vartheta\,\boldsymbol{\xi}^a\,\boldsymbol{\xi}^b\,\Gamma^b_{AB}\,, \tag{6.54}$$

and

$$\bar{\eta}_i \Gamma^{ab} \eta_j \quad \longmapsto \quad \Gamma^{ab}_{AB} + 2\,i \sin(2\vartheta)\,\boldsymbol{\xi}^{[a}\Gamma^{b]}_{AB} + 4\sin^2\vartheta\,\boldsymbol{\xi}^c\boldsymbol{\xi}^{[a}\,\Gamma^{b]c}_{AB}\,. \tag{6.55}$$

Note that, in particular, we have

$$\xi^a\,\bar{\eta}_i\Gamma^a\eta_j \quad \longmapsto \quad \xi^a\,\Gamma^a_{AB}\,, \tag{6.56}$$

as expected from a rotation about ξ^a.

After tedious algebra, we find

$$\mathcal{A}_{m\,ABCD}(\alpha,\beta) = P_1\,\boldsymbol{\xi}_m\,\Gamma^a_{[AB}\Gamma^a_{CD]} + P_2\,\boldsymbol{\xi}^a\,\Gamma_{m[AB}\Gamma^a_{CD]} + P_3\,\boldsymbol{\xi}_m\boldsymbol{\xi}^a\boldsymbol{\xi}^b\,\Gamma^a_{[AB}\Gamma^b_{CD]}$$
$$+\,Q_1\,i\,\Gamma_{m[AB}^a\Gamma^a_{CD]} + Q_2\,i\,\boldsymbol{\xi}_m\boldsymbol{\xi}^a\,\Gamma^{ab}_{[AB}\Gamma^b_{CD]}\,, \tag{6.57}$$

which has the correct form (3.18) of a solution to the GVP in $d = 11$. Indeed, the first three terms on the right hand side, with

$$\frac{P_1}{P_0} = \tfrac{2}{3}\,H^2\,, \qquad \frac{P_2}{P_0} = -2H\,, \qquad \frac{P_3}{P_0} = -2H(H-1)\,, \tag{6.58}$$

and the common factor

$$P_0 = -\tfrac{3}{4}\,(\alpha+4\beta)\,\frac{\tau(1+9\tau)}{(1+21\tau)^2}\,\sqrt{(21+\xi)(3-\xi)}\,, \tag{6.59}$$

combine correctly to

$$-\tfrac{3}{4}(\alpha+4\beta)\,(S^{-1}\mathring{D}_m S)_{(ab)}\,\Gamma^a_{[AB}\Gamma^b_{CD]}\,, \tag{6.60}$$

confirming the relation $\alpha+4\beta = 1$. This can be checked by evaluating the background derivative of the siebenbein (6.46) using two identities

$$\mathring{D}_m\xi = 2\sqrt{(21+\xi)(3-\xi)}\,\boldsymbol{\xi}_m\,,$$
$$\mathring{D}_m\boldsymbol{\xi}^a = \sqrt{\frac{3-\xi}{21+\xi}}\,(\mathring{e}_m{}^a - \boldsymbol{\xi}_m\boldsymbol{\xi}^a)\,, \tag{6.61}$$

that follow from the definition (6.40).

The coefficients in the remaining two terms in (6.57) are considerably more complicated:

$$Q_1 = -\tfrac{1}{8}\,(\alpha-4\beta)\frac{1+21\tau}{1-3\tau}\,\frac{1}{H} + \tfrac{1}{8}\,(\alpha+4\beta)\,\frac{1-3\tau}{1+21\tau}\,H\,, \tag{6.62}$$

and

$$Q_2 = \beta\,\frac{1+21\tau}{1-3\tau}\,\frac{1}{H^2} + \tfrac{1}{8}(\alpha-4\beta)\frac{1+21\tau}{1-3\tau}\,\frac{1}{H}$$
$$-\tfrac{1}{4}\,(\alpha+4\beta)\frac{1+18\tau+225\tau^2}{(1-3\tau)(1+21\tau)}$$
$$-\tfrac{1}{8}\,(\alpha+4\beta)\frac{1-3\tau}{1+21\tau}\,H + \tfrac{1}{4}(\alpha+4\beta)\frac{1-3\tau}{1+21\tau}\,H^2\,. \tag{6.63}$$

Comparing (6.57) with (3.18), we read off X_{ma} and confirm that $X_{m\,abc} = 0$. By contracting the former with the inverse siebenbein, see (3.19), and setting α and β to the required values (2.21), we find that

$$X_{a|b} = X_0\,\delta^{ab} + X_2\,\boldsymbol{\xi}^a\boldsymbol{\xi}^b\,, \tag{6.64}$$

where

$$
\begin{aligned}
X_0 &= -\frac{(21\tau + 1)^2 - 7H^2(1 - 3\tau)^2}{56H^{4/3}(1 - 3\tau)^{7/6}(21\tau + 1)^{5/6}}\,,\\
X_2 &= \frac{(H^2 - 1)\,(2H^2(1 - 3\tau)^2 - (21\tau + 1)^2)}{8H^{4/3}(1 - 3\tau)^{7/6}(21\tau + 1)^{5/6}}\,.
\end{aligned}
\tag{6.65}
$$

Decomposing $X_{a|b}$ into irreducible components (3.20), we get

$$X = X_0 + \tfrac{1}{7}\,X_2\,, \tag{6.66}$$

$$X_{ab}^{\boxplus} = -\tfrac{1}{7}\,X_2\,\delta_{ab} + X_2\,\boldsymbol{\xi}_a\boldsymbol{\xi}_b\,. \tag{6.67}$$

From (5.2) we now obtain the flux along the entire $\mathrm{SO}(7)^+$ family,

$$f = \frac{\sqrt{2}\,H^{2/3}\,(H^2(1 - 3\tau)^2 - 198\tau^2 - 36\tau + 2)}{(1 - 3\tau)^{7/6}(21\tau + 1)^{5/6}}\,, \qquad F_{abcd} = 0\,. \tag{6.68}$$

Once more, at the $\mathrm{SO}(8)$ point with $\tau = 0$ and $H = 1$, we reproduce (6.2). At the $\mathrm{SO}(7)^+$ point, we have $99\tau^2 + 18\tau = 1$, and therefore the expression for f simplifies to

$$f = 2^{1/2} \cdot 5^{3/4}\,\Delta^2\,, \tag{6.69}$$

which agrees with the known solution, see Table I in [26].[15] As required by consistency, the Freund-Rubin parameter $f_0 = f\Delta^{-2}$ becomes y-independent at the critical point. Hence the standard inhomogeneous solution does reproduce the correct flux at the $\mathrm{SO}(7)^+$ point as well.

However, since X_2 does not vanish outside the maximally supersymmetric point, we conclude that in this example *the standard inhomogenous solution does not have the correct tensor structure* (2.12). This can be seen even more directly by comparing the last two terms in (6.57) with the $e_{ma}f$ term in (2.12). In order that the matrix

$$Q_1\mathring{e}_{ma} + Q_2\,\boldsymbol{\xi}_m\boldsymbol{\xi}_a\,, \tag{6.70}$$

be *proportional* to e_{ma}, we must have

$$Q_2 - \left(\frac{1}{H} - 1\right)Q_1 = 0\,. \tag{6.71}$$

---

[15]The comparison involves setting the same overall normalization of the solutions, see Section 7.4 below.

Evaluating the left hand side we get

$$(\alpha + 4\beta)\frac{(H^2 - 1)\left(2H^2(1 - 3\tau)^2 - (21\tau + 1)^2\right)}{8H^2(3\tau - 1)(21\tau + 1)} = 0\,, \tag{6.72}$$

which can vanish at each point on S^7 when either $H \equiv 1$, which is the maximally supersymmetric solution, or when $\alpha + 4\beta = 0$. However, the latter cannot be satisfied given (2.21). In the next section we will argue that this appears to be a generic feature of the $\mathcal{A}_{m\,ijkl}(\alpha, \beta)$ tensor in (2.20). To see that there is no discrepancy here at all, we verify explicitly that the standard inhomogeneous solution can be shifted as in Section 4, such that one obtains a solution satisfying the correct tensor structure condition. From the general solution to the homogeneous equations found in Sections 3 and 4, and using the $SO(7)^+$ invariance to limit the allowed α-parameters, we find that the correction must be of the form

$$\delta\,\mathcal{A}_{m\,ijkl} = \left(\alpha_m{}^n + \tfrac{3}{7}\,\delta_m{}^n\right)\mathcal{A}_{n\,ijkl}^{(\alpha)} - \left(\tfrac{1}{4}\,\alpha_m{}^n + \tfrac{3}{28}\,\delta_m{}^n\right)\mathcal{A}_{n\,ijkl}^{(\beta)}\,. \tag{6.73}$$

The result for the U-rotated tensor can be read-off from the α and β components of $\mathcal{A}_{m\,ABCD}(\alpha, \beta)$ in (6.57). We then find that the $S^{-1}\mathring{D}_m S$ terms cancel, as they should. Imposing the correct tensor structure condition yields a unique solution

$$\alpha_m{}^n(\xi, \tau) = a_0(\xi, \tau)\,\delta_m{}^n + a_2(\xi, \tau)\,\boldsymbol{\xi}_m\boldsymbol{\xi}^n\,, \tag{6.74}$$

where

$$a_0 = -\tfrac{1}{7}\,a_2 = -\frac{(H^2 - 1)\left(2H^2(1 - 3\tau)^2 - (21\tau + 1)^2\right)}{14(21\tau + 1)^2}\,. \tag{6.75}$$

Note that $\alpha_m{}^m = 0$ as required, however, the correction breaks the $SO(7)$ 'background covariance' in the sense of the comment after (3.14).

7   Numerical examples

In this section, we summarize some numerical tests that led us to reexamine the proof of the consistent truncation in [3], and discuss the standard inhomogenous solution and fluxes for additional critical points. We also found 'numerical explorations' to be quite helpful in developing analytic arguments in Sections 3-5. The term 'numerical' is used here in a wide sense; it means either an actual numerical solution to the system of equations given by the consistent truncation ansätze, or simply an explicit evaluation (mixed numerical and analytic) of expressions using specific coordinates on the internal manifold and a particular representation of the internal Γ-matrices.

It is important to note that the construction of the lift for the metric and the fluxes at a given point on the scalar manifold, $E_{7(7)}/SU(8)$, is 'algebraic' with respect to the internal space. This is of course manifest for the metric, cf. (2.9), but is also true for the fluxes, the simple reason being that the background covariant derivative always acts on the Killing spinors or vectors on S^7 and, by virtue of (2.7) or (6.22), respectively, is effectively an algebraic operation. Hence, all equations given by the ansätze can be evaluated and then solved at each point on S^7 independently, and the solution involves only algebraic operations.

7.1 Preliminaries

In the following, we use stereographic coordinates on S^7 and the Killing spinors that are obtained as follows:[16] Represent S^7 as the surface, $(X^1)^2 + \ldots + (X^8)^2 = m_7^{-2}$, in $\mathbb{R}^8$. The stereographic coordinates, $y^a \in \mathbb{R}$ $(a = 1, \ldots, 7)$, are defined by

$$X^i = m_7^{-1} \frac{2 y^i}{1 + (y^a)^2}, \qquad X^8 = m_7^{-1} \frac{1 - (y^a)^2}{1 + (y^a)^2}, \qquad i = 1, \ldots, 7, \tag{7.1}$$

and we use

$$\mathring{e}^a = -\frac{2\, m_7^{-1}\, dy^a}{1 + (y^a)^2}, \qquad a = 1, \ldots, 7, \tag{7.2}$$

as the background siebenbein. Then the spin-connection 1-forms are $\mathring{\omega}^{ab} = -m_7\, (y^a \mathring{e}^b - y^b \mathring{e}^a)$, such that the Ricci tensor is $\mathring{R}_{ab} = 6\, m_7^2\, \delta_{ab}$.

The matrix, $\eta = (\eta^I{}_A)$, of the Killing spinors is

$$\eta(y) = \frac{\mathbf{1} + i\, y^a\, \Gamma^a}{\sqrt{1 + (y^b)^2}} . \tag{7.3}$$

It is easy to check that η^I's satisfy the Killing spinor equation (2.7), and that η is a real orthogonal matrix. At the North Pole, $y^a = 0$, we have $\eta^I{}_A = \delta^I{}_A$ and $\mathring{\omega}^{ab} = 0$.

We use the SO(7) gamma matrices, Γ^a_{AB}, $a = 1, \ldots, 7$, that are antisymmetric and purely imaginary and satisfy

$$\Gamma^a \Gamma^b + \Gamma^b \Gamma^a = 2\, \delta^{ab}\, \mathbf{1} , \tag{7.4}$$

$$\Gamma^7 = i\, \Gamma^{123456} . \tag{7.5}$$

If needed, explicit representations with these properties can be found in Appendix C.1 of [46], or in Appendix C of [28]. Note that the latter is for the negative Euclidean signature and gives the opposite sign in (7.5).

---

[16]For a systematic discussion, see [45] and the references therein.

7.2 Initial numerical tests

In our initial tests, we looked at the orginal flux formula (2.26) for two non-supersymmetric critical points of the scalar potential of $\mathcal{N}=8$, $d=4$ supergravity: the perturbatively unstable $\mathrm{SU}(4)^-$ point and the perturbatively stable $\mathrm{SO}(3)\times\mathrm{SO}(3)$ point. The same calculation performed for a random scalar 56-bein yields similar results.

Starting with (2.26), which is (7.5) in [3], we evaluate the trace over $m=q$. Since the flux F_{mnpq} should be totally antisymmetric, we then get

$$\tfrac{12}{7}(if)\,g_{np} = -i\tfrac{\sqrt{2}}{480}\,\Delta^4\,\epsilon_{pqrstuv}\,e^q_{ij}(e^r\bar{e}^s e^t \bar{e}^u e^v)_{kl}\mathcal{A}_n{}^{ijkl}(\alpha,\beta)\,. \tag{7.6}$$

The left hand side is now proportional to the metric tensor, and thus its contraction with the inverse metric tensor $\Delta^{-1}g^{mn}$ in (2.9) should yield a result proportional to the identity matrix. To test that, we would fix a point on S^7, either at the North Pole or at some random value of the stereographic coordinates, and evaluate numerically:

 (i) the inverse tensor in (2.9), and

 (ii) the tensor defined by the right hand side in (7.6) for arbitrary values of α and β.

The result is that for the $\mathrm{SU}(4)^-$ point, the contraction between the tensors (i) and (ii) is not proportional to the unit matrix, while for the $\mathrm{SO}(3)\times\mathrm{SO}(3)$ point, the tensor in (ii) is not even symmetric. In both examples, the undesired terms are proportional to $\alpha+4\beta$ and do *not* vanish for the standard inhomogenous solution (2.21). In retrospect, they arise because $\mathring{\mathcal{A}}_{m\,ijkl}$ does not satisfy the correct tensor structure condition (2.12).

7.3 Tensor structure tests

The main purpose of a more systematic numerical exploration is to determine the structure of the $\mathring{\mathcal{A}}_{m\,ABCD}$ tensor obtained by the U-rotation of the $\mathring{\mathcal{A}}_{m\,ijkl}$ tensor of the standard inhomogenous solution.

Once more we take a scalar 56-bein, $\mathcal{V}$, for one of the critical points and choose a random point on S^7. We then evaluate numerically the metric tensor, g_{mn}, and the warp factor, Δ, using (2.9). By taking the matrix square root of the metric tensor, we find the metric vielbein, $e_m{}^a$, and its inverse. Then (2.8) becomes a quadratic equation for the $\mathrm{SU}(8)$-matrix, U, with the latter determined up to an overall sign that cancels in (2.23). The resulting $\mathring{\mathcal{A}}_{m\,ABCD}$ tensor is expanded in the canonical basis, cf. (3.18), and we read-off the expansion coefficients, $X_{m\,a}$ and $X_{m\,abc}$. Finally we evaluate $X_{a|b}$ and $X_{a|bcd}$ in (3.19) and decompose them into irreducible $\mathrm{SO}(7)$ components (3.20).

Our results are summarized in Table 2, where the star indicates that a given irreducible component does not vanish. The critical points are listed in the first column using the labelling

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$\mathcal{V}$	Symmetry	X	$X^{\boxed{}}_{abcd}$	$X^{\boxed{}}_{ab}$	$X^{\boxed{}}_{ab}$	$X^{\boxed{}}_{abcd}$	$\delta_{a[b}\tilde{X}^{\boxed{}}_{cd]}$
S0600000	SO(8)	*					
S0668740	SO(7)$^-$	*	*				
S0698771	SO(7)$^+$	*		*			
S0719157	G$_2$	*	*	*	*	*	*
S0779422	SU(3) $\times$ U(1)	*	*	*	*	*	*
S0800000	SU(4)$^-$	*	*	*			
S0880733	SO(3) $\times$ SO(3)	*	*	*	*	*	*
S1200000	U(1) $\times$ U(1)	*	*	*	*	*	*
S1400000	SO(3) $\times$ SO(3)	*	*	*	*	*	*

Table 2: Irreducible components in $\mathcal{A}_{m\,ABCD}\big(\tfrac{4}{7},\tfrac{3}{28}\big)$ at some critical points

scheme in [20] that is based on the value of the cosmological constant. The second column gives the symmetry of each point, which is perhaps more recognizable than the label. The reader may consult [20] for additional information about and references for each point. The last four columns are the components that violate the tensor structure condition: we see that already the highly symmetric G$_2$ solution gives rise to all possible tensor structures. This table includes all 'old' critical points and two 'new' ones, S0880733 and S1200000, first found numerically in [18, 20] and then further investigated in [19].

7.4   Critical points

The calculation in Section 7.3 also gives us the flux, f, which can be readily compared with the known lifts of critical points. Recall that at a critical point, the flux, f, along AdS_4 is by conformal invariance and the Bianchi identity of the form $f = f_0\Delta^2$, where f_0 is a constant. We determine numerically the value of f_0 from the coefficient X_{ma} in $\mathring{\mathcal{A}}_{m\,ABCD}$ using (5.10). Then we verify that f_0 is indeed constant by performing the same calculation for two or more points on the sphere. The results are listed in Table 3.

Next we compare our numerical results with the known solutions. This requires ensuring that a solution (1.1)-(1.2) obtained from the lift and a solution we compare it with have the same overall normalization. The potential mismatch between the normalizations comes from the fact that the field equations of $d=11$ supergravity are invariant under the rescaling

$$g_{MN} \to \lambda\,g_{MN}\,, \qquad F_{MNPQ} \to \lambda^{3/2} F_{MNPQ}\,, \tag{7.7}$$

where λ is a constant. This rescaling preserves the form of a solution (1.1)-(1.2), but changes the radius of AdS_4 and the overall scale of the internal metric and the flux. Let us also note that

$\mathcal{V}$	Symmetry	$-\mathcal{P}_*/g^2$	f_0/m_7	Refs
S0600000	SO(8)	6	4.242641	[23]
S0668740	SO(7)$^-$	6.68740	4.941059	[43, 39]
S0698771	SO(7)$^+$	6.98771	4.728708	[44, 26]
S0719157	G_2	7.19157	5.085212	[26]
S0779422	SU(3) $\times$ U(1)	7.79422	5.511352	[29]
S0800000	SU(4)$^-$	8	5.656854	[47]
S0880733	SO(3) $\times$ SO(3)	8.80733	6.227729	–
S1200000	U(1) $\times$ U(1)	12	8.485281	–
S1400000	SO(3) $\times$ SO(3)	14	9.899495	[30]

Table 3: The f_0-flux at some critical points.

the relative normalization between the AdS_4 and the internal parts of the metric and the flux is completely fixed by the equations of motion. In particular, from a linear combination of the Einstein equations, we have[17]

$$R_m{}^m + \tfrac{5}{4}\, R_\mu{}^\mu = f_0^2 \Delta^4 \, , \tag{7.8}$$

which effectively is the equation of motion we are testing here.

The solution obtained from the lift comes with a particular normalization determined by the explicit embedding of the $d=4$ solution in eleven dimensions. Specifically, the radius, L, of AdS_4 in (1.1) is given by

$$L^2 \equiv m_4^{-2} = -\frac{3}{\mathcal{P}_\star} \, , \tag{7.9}$$

where $\mathcal{P}_*$ is the value of the scalar potential of the $\mathcal{N}=8$ theory [1],

$$\mathcal{P} = -g^2 \left(\tfrac{3}{4} \left| A_1{}^{ij} \right|^2 - \tfrac{1}{24} \left| A_{2i}{}^{jkl} \right|^2 \right) , \tag{7.10}$$

at the critical point. The normalization of the internal metric and the flux are in turn determined by the lift formulae (2.9) and (5.2).

To test the first four points, we use solutions summarized in Table I in [26], where we find

$$m_4^2 = a\, m_7^2\, \gamma^{1/2} \, , \qquad f_0 = b\, m_7\, \gamma^{5/6} \, . \tag{7.11}$$

The values of the constants a and b depend on the critical point under consideration and can be read off from Table I in [26], while γ is an arbitrary parameter that sets the overall normalization of the solution. For each critical point we find the correct γ by solving (7.9) with m_4 in (7.11).

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[17]See, e.g. (3.7) in [26]. As shown there, this equation implies the relation $15m_4^2\gamma^{-1/2} - f_0^2\gamma^{-5/3} = 42m_7^2$, explaining the powers of γ appearing in (7.11).

Then we use this particular value of γ to evaluate f_0 in (7.11) and in all four cases find a complete agreement with the numerical values in Table 3.

The $SU(3) \times U(1)$ solution is given in [29]. From (4.29) and (4.33) in that paper we get[18]

$$f_0 = \frac{3^{7/4}}{2^{3/2}} \frac{1}{L}. \tag{7.12}$$

At the critical point,

$$\mathcal{P}_* = -\frac{3^{5/2}}{2} g^2, \tag{7.13}$$

so from (7.9), and recalling that $g = \sqrt{2}\, m_7$, we get

$$\frac{1}{L} = 3^{3/4}\, m_7. \tag{7.14}$$

Substituting this in (7.12) yields

$$f_0 = \frac{3^{5/2}}{2^{3/2}}\, m_7 \approx 5.51135\, m_7, \tag{7.15}$$

which agrees with the numerical value in Table 3.

The solution at the $SU(4)^-$ critical point was found in [47]. For the comparison we use the explicit formulae (4.70) and (4.71) in [35], which after rescaling the flux (see, footnote 18) read

$$ds_{11}^2 = ds_{AdS_4}^2 + \dots, \qquad F_{(4)} = \sqrt{\frac{3}{2}} \frac{1}{L} \operatorname{vol}_{AdS_4} + \dots. \tag{7.16}$$

However, the metric obtained from the lift (2.9) has a nonvanishing constant warp factor, $\Delta = 2^{-2/3}$. Reintroducing this warp factor in (7.16) by rescaling the metric by Δ^{-1} and the flux by $\Delta^{-3/2}$, we get $f_0 = \sqrt{6}/L$. Then, using $\mathcal{P}_* = -8g^2$, and normalizing the AdS_4 radius according to (7.9), we get

$$f_0 = 4\sqrt{2}\, m_7 \approx 5.65685\, m_7, \tag{7.17}$$

which is the same as the numerical value obtained from the lift.

Finally, the flux, f_0, for the $SO(3) \times SO(3)$ critical point has been calculated in [30] by solving (7.8), where the metric is obtained from the lift formula (2.9). Once more we find that it agrees with the numerical result in Table 3. Analytic solutions for the remaining two points in Table 3 are not known explicitly in closed form.

In Table 3 we have also listed the values of the scalar potential at each of the critical points. We see that there is a universal relation between $\mathcal{P}_*$, or equivalently the radius of AdS_4, and f_0, which in our normalization reads

$$-\frac{\mathcal{P}_*}{g^2} = \sqrt{2}\, \frac{f_0}{m_7}. \tag{7.18}$$

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[18]There is a difference in the normalization of the flux in [29] and in this paper, $F_{(4)} = \sqrt{2}\, F_{(4)}^{\text{CPW}}$.

In principle, this relation is a consequence of (7.8), but we are not aware of any simple proof of it. The difficulty here is that the components of the Ricci tensor in (7.8) are for the full $d = 11$ metric.

Curiously, the relation (7.18) holds for some other field configurations, for instance in the entire $\mathrm{SO}(7)^-$ invariant sector, see (6.13), (6.18) and (6.25). However, it is not valid in general, in particular, at a generic point in the $\mathrm{SO}(7)^+$ sector, where $f\Delta^{-2}$, c.f. (6.48) and (6.68), has a nontrivial dependence on the sphere coordinates which cancels out only at the two critical points.

While our tests in this section were limited only to solutions corresponding to the critical points, and we looked only at the flux component along AdS_4, it is clear that the agreement we have found is a striking confirmation of the lift formulae for the flux.

8   Conclusions and outlook

In this paper we have clarified the structure of the equations given in [3] (that is, the *GVP* and the $\mathfrak{A}$-equations) which characterize consistent truncations of eleven-dimensional supergravity on $AdS_4 \times S^7$ to gauged supergravity. We have revealed a hidden degeneracy in these equations and demonstrated that this degeneracy is precisely what is needed in order to remove apparent discrepancies arising in the comparison between the $d = 4$ and $d = 11$ expressions, and to recover the correct tensor structure of the fluxes required by the $d = 11$ theory for *any* given non-trivial solution of the $d = 4$ theory. Furthermore, we have clarified the status of the non-linear ansätze for the fluxes, and shown that these constitute invariants of the consistency equations.

These 'flux lift formulae' can now be put to practical use, and we have presented several non-trivial tests, both analytic and numerical. It is also clear from our discussion that on the one side an analytic calculation of the fluxes based on those formulae is quite difficult and cumbersome, though perhaps it can be simplified in particular examples by a judicious choice of coordinates and the Killing vectors/spinors. On the other side, a numerical calculation is reasonably straightforward and very likely may be sufficient to determine properties of the full solutions one might be interested in.

Let us also remark that the degeneracy problem discussed in this paper does not arise for the 'mixed' flux components: $F_{\mu bcd}$, $F_{\mu\nu bd}$, etc., which will no longer vanish for x-dependent solutions of the $d = 4$ theory; these can therefore be determined unambiguously from the corresponding formulas given in [27] and [3, 4].

Finally, our results may also be relevant in the context of the $AdS_5 \times S^5$ compactification of IIB supergravity, for which the analog of the metric lift formula (2.9) is known, but a complete proof of the consistency is still lacking. *Mutatis mutandis* we anticipate that the techniques developed here on the basis of [3] will also apply to this case.

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